Department of Mathematics

The Transmission Eigenvalue Problem, Non-scattering Phenomena and Inverse Scattering for Inhomogeneous Media

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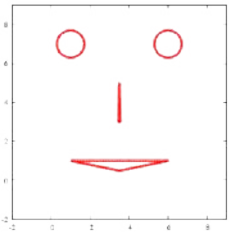
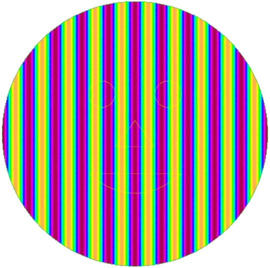
RUTGERS

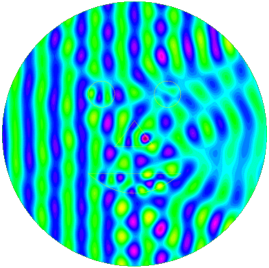
Scattering of Waves



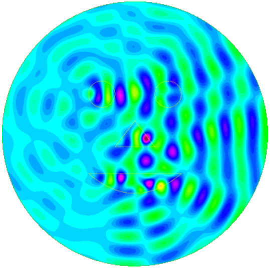
Scattering Media



Probing (incident) Field



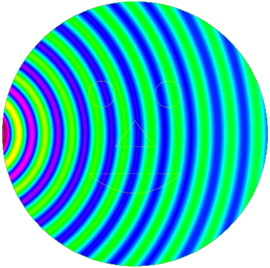
Total Field



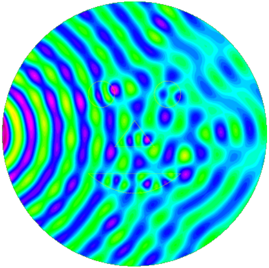
Scattered Field

Scattering of Waves

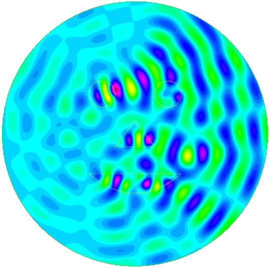
Scattering Media



Probing (incident) Field



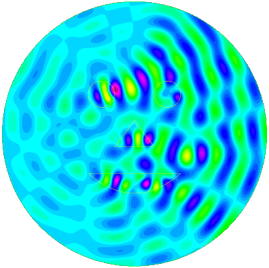
Total Field

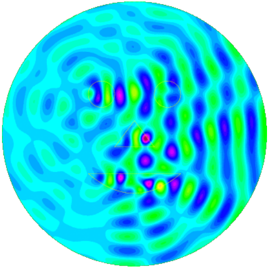


Scattered Field

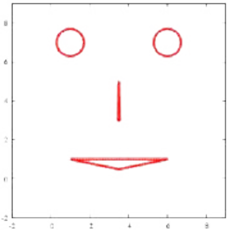
Inverse Scattering – Imaging with Waves

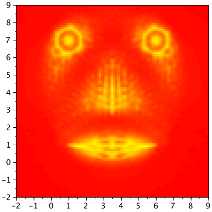
From the knowledge of scattered field





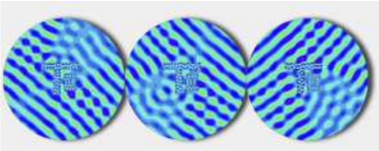
reconstruct the perturbation

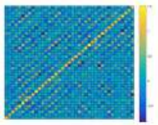




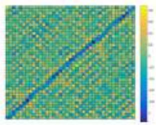
Imaging Method

Typically measurements due to various probing is used:



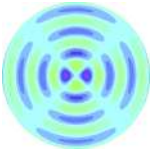


Far Field Real Part

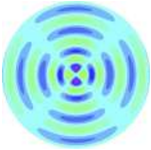


Far Field Imaginary Part

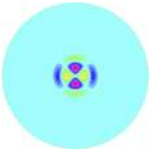
Appropriate superposition of measurements leads to an imaging method.
In the frequency domain, certain superpositions lead to non-scattering.



incident field



total field



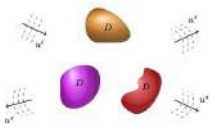
scattered field

Basic Literature

- D. Colton and R. Kress (2019), *Inverse Acoustic and Electromagnetic Scattering Theory*, 4th Edition, Springer.
- A. Kirsch - N. Grinberg (2008), *The factorization method for inverse problems*, Oxford University Press.
- A. Kirsch (2022), *An introduction to the mathematical theory of inverse problems*, 3rd Edition Springer.
- F. Cakoni, D. Colton and H. Haddar (2023), *Inverse Scattering Theory and Transmission Eigenvalues*, CBMS-NSF, SIAM Publication, 2nd Edition.

- F. Cakoni, D. Colton and H. Haddar (2021) *Transmission Eigenvalues*, AMS Notices, October Issue.

Scattering by an Inhomogeneous Media



∂D is Lipschitz, $k = \omega / c_b$, $\rho_b = 1$

$n \in L^\infty(\mathbb{R}^3)$, $n = n_1 + \frac{i}{k} n_2$, $n_1 > 0$ and $n_2 \geq 0$.

$$\text{Supp}(n - 1) \qquad \text{is bounded}$$

- The incident field v satisfies the Helmholtz equation

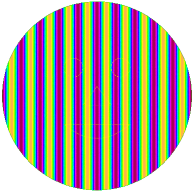
$$\Delta v + k^2 v = 0 \qquad \text{in } \mathbb{R}^3$$

- The total field $u = w + v$ satisfies

$$\Delta u + k^2 n u = 0 \qquad \text{in } \mathbb{R}^3$$

- The scattered field w is outgoing, i.e. it satisfies the Sommerfeld radiation condition

Scattering by an Inhomogeneous Media



Take plane incident wave

$$v = u^i(x; \hat{y}, k) := e^{ikx\cdot\hat{y}}$$

- $u := u(x; \hat{y}, k)$ corresponding total field
- $w := u^s(x; \hat{y}, k)$ corresponding scattered field

The outgoing scattered field satisfies

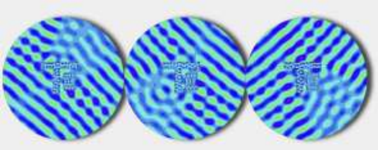
$$w(x) = \frac{e^{ik|x|}}{|x|} w^\infty(\hat{x}) + O\left(\frac{1}{|x|^2}\right) \qquad \text{as } |x| \to \infty, \;\; \hat{x} = x/|x|$$

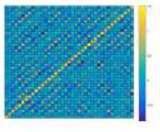
$w^\infty(\hat{x})$ defined on the unit sphere $\mathbb{S}^2$ is called the far field pattern.

(Rellich's Lemma)

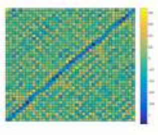
$$w^\infty(\hat{x}) = 0 \;\; \forall\, \hat{x} \in \mathbb{S}^2 \implies w(x) = 0 \;\; \forall\, x \in \mathbb{R}^3 \setminus \overline{D}$$

Far Field Operator





Far Field Real Part



Far Field Imaginary Part

Far Field Operator (aka Relative Scattering Operator

$$F_k : L^2(\mathbb{S}) \to L^2(\mathbb{S}), \qquad (F_k g)(\hat{x}) := \int_{\mathbb{S}} g(\hat{y}) u^{\infty}(\hat{x}; \hat{y}, k)\, ds$$

$Fg = w_g^{\infty}$ where w_g^{∞} is the far field pattern of the scattered field w_g

with incident wave $v_g(x) := \int_{\mathbb{S}} g(\hat{y}) e^{ikx\cdot\hat{y}}\, d\hat{y}$ Herglotz wave function

Far Field Operator

Is the far field operator $F_k : L^2(\mathbb{S}) \to L^2(\mathbb{S})$ injective?

KIRSCH,(1986) COLTON-MONK (1987)

$F_k g = 0$ if and only if there exists a Herglotz function $v_g(x) := \int_{\mathbb{S}} g(\hat{y}) e^{ikx\cdot\hat{y}}\, d\hat{y}$ such that $w_g^\infty = 0$, hence the corresponding scattered field $w_g = 0$ outside D.

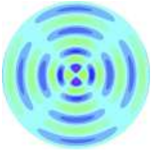
Definition

$k > 0$ is called non-scattering wave number for given inhomogeneity n, D if F_k is not injective.

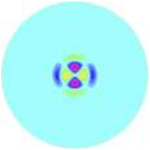
An inhomogeneity n, D that admits a non-scattering wave number is referred to as non-scattering inhomogeneity, and v_g for which $Fg = 0$ as non-scattering incident wave.

Non-scattering Inhomogeneity ($*$)

incident field



total field



scattered field

Simple calculations show that $F^*g = \overline{RFR\overline{g}}$ where $(Rh)(\hat{x}) := h(-\hat{x})$.
Thus we have proven:

Theorem

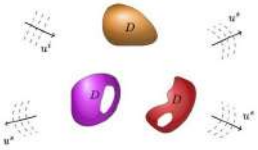
The far field operator $F_k : L^2(\mathbb{S}) \to L^2(\mathbb{S})$ is injective and has dense range if and only if $k > 0$ is not a non-scattering wave number.

Scattering by an Inhomogeneous Media

The scattered field $w \in H^2_{loc}(\mathbb{R}^3)$ due to a Herglotz wave function v_g satisfies

$$\Delta w + k^2 n w = k^2(1-n)v_g \qquad \text{in } \mathbb{R}^3 \qquad \text{plus SRC}$$

and of course we have $\Delta v_g + k^2 v_g = 0 \qquad$ in $\mathbb{R}^3$



$$\overline{\mathcal{O}} = \text{Supp}(n-1)$$

Let G be the unbounded component of $\overline{O}^c$
We call $D := \overline{G}^c$, and $\overline{D} \subset \Omega$
Now assume that the incident field v_g does not scatter, this is

$$w \equiv 0 \quad \text{in} \quad \mathbb{R}^3 \setminus \overline{D}$$

The equation for w implies $\qquad w = \dfrac{\partial w}{\partial \nu} = 0 \qquad$ on ∂D

Non-scattering inhomogeneity

(Non-scattering inhomogeneity)

Given the inhomogeneous media (n, D), we say k is a non-scattering wave number if the following problem has solution

$$\Delta w + k^2 n w = k^2(1-n)v_g \qquad \text{in } D$$

$$w = \frac{\partial w}{\partial \nu} = 0 \qquad \text{on } \partial D$$

with $v_g(x) = \int_{\mathbb{S}} g(\hat{y}) e^{ikx\cdot\hat{y}} ds$, satisfying $\Delta v_g + k^2 v_g = 0 \qquad \text{in } \mathbb{R}^3$

Over-determined problem!!!

To mitigate this, instead of v_g we consider $L^2(D)$ (distributional) solutions to the equation

$$\Delta v + k^2 v = 0 \qquad \text{in } D$$

The transmission Eigenvalue Problem

(Transmission eigenvalue problem for w and v)

Given (n, D), we say k is a transmission eigenvalue if the following problem has non-trivial solution $w \in H_0^2(D)$ and $v \in L^2(D)$

$$\begin{aligned} \Delta w + k^2 n w &= k^2(1-n)v \qquad \text{in } D \\ \Delta v + k^2 v &= 0 \qquad \text{in } D \end{aligned}$$

$$w = \frac{\partial w}{\partial \nu} = 0 \qquad \text{on } \partial D$$

(Transmission eigenvalue problem for $u := w + v$ and v)

Find nonzero $u \in L^2(D)$ and $v \in L^2(D)$, such that $u - v \in H^2(D)$ satisfying

$$\Delta v + k^2 v = 0 \quad \text{and} \quad \Delta u + k^2 n u = 0 \quad \text{in } D$$

$$u - v = 0 \quad \text{and} \quad \frac{\partial u}{\partial \nu} - \frac{\partial v}{\partial \nu} = 0 \quad \text{on } \partial D$$

The Transmission Eigenvalue Problen

- Non-scattering wave numbers are subset of real transmission eigenvalues.
- A real transmission eigenvalue is non-scattering wave number if part v of the eigenfunction is a Herglotz wave function, (more generally is extendable as solution to the Helmholtz equation in $\mathbb{R}^3$).
- A non-scattering wave number is related to a scattering experiment. However, the existence of non-scattering wave numbers generically implies certain regularity of D and n.

We will study the Transmission Eigenvalue Problem.

Spherical Geometry

$D := B_1(0)$, $n(r)$ real valued positive, and separate variables

$$v = \sum_{\ell}^{\infty} a_\ell j_\ell(k|x|) Y_\ell(\hat{x}), \qquad u = \sum_{\ell}^{\infty} u_\ell(r) Y_\ell(\hat{x})$$

where $Y_\ell(\hat{x})$ denotes $2\ell + 1$ spherical harmonics of order $\ell \in \mathbb{N}$ which all together form a Fourier basis in $L^2(\mathbb{S})^2$, $j_\ell(r)$ are spherical Bessel functions and $u_\ell(r) := u_\ell(r; k, n)$ solves (regular at $r = 0$) Bessel Equation

$$z'' + \frac{2}{r} + \left(k^2 n(r) - \frac{\ell(\ell+1)}{r^2}\right) z = 0.$$

Applying the boundary conditions at $r = 1$ gives that all transmission eigenvalues are the zeros

$$d_\ell(k) = \mathrm{Det} \begin{pmatrix} u_\ell(1; k, n) & j_\ell(k) \\ u'_\ell(1; k, n) & k j'_\ell(k) \end{pmatrix} = 0$$

Spherically Symmetric Media

- All real transmission eigenvalues are non-scattering wave numbers since $v_\ell = j_\ell(k|x|)Y_\ell(\hat{x})$ is a Herglotz wave functions.
- The only non-scattering incident waves are $v_\ell = j_\ell(k|x|)Y_\ell(\hat{x})$.
- Each v_ℓ is not scattered at an infinite set of wave numbers $k > 0$ with accumulation point $+\infty$
- Thus, the scattering operator F_k is non-injective at an infinite countable set of wave numbers.

D. Colton and R. Kress (2019), *Inverse Acoustic and Electromagnetic Scattering Theory*, Springer, 4rth Edition.

Spherically symmetric configuration is unstable.

Michael Vogelius and Jingni Xiao (2021), *Finiteness results concerning non-scattering wave numbers for incident plane and Herglotz waves*, SIAM J. Math Analysis.

TE for Spherically Symmetric Media

Consider radially symmetric eigenfunction i.e. for $\ell = 0$, thus

$$v(x) := a_0 j_0(kr) = a_0 \frac{\sin kr}{kr} = \frac{a_0}{4\pi} \int_{\mathbb{S}} e^{ikx\cdot\hat{y}} ds_y \qquad \text{and} \qquad u(x) = b_0 \frac{y(r)}{r}$$

$$\text{and} \qquad y'' + k^2 n(r) y = 0 \qquad y(0) = 0,\ y'(0) = 1$$

$$d(k) = \begin{vmatrix} y(1; k, n) & \dfrac{\sin k}{k} \\ y'(1; k, n) & \cos k \end{vmatrix} = 0$$

To understand the solution $y(r)$ we use the Liouville transformation

$$\xi(r) := \int_0^r \sqrt{n(\rho)}\, d\rho \qquad z(\xi) := n(r)^{1/4} y(r), \text{ at } r = r(\xi)$$

to arrive at

$$z'' + (k^2 - p(\xi)) z = 0 \qquad z(0) = 0,\ z'(0) = n(0)^{-1/4}$$

$$p(\xi) = \frac{n''(r)}{4n(r)^2} - \frac{5}{16} \frac{n'(r)^2}{n(r)^3}$$

TE for Spherically Symmetric Media

The problem for z can be written as Volterra integral equation

$$z(\xi) = \frac{\sin k\xi}{k n(0)^{1/4}} + \frac{1}{k}\int_0^{\xi} \sin k(\eta - \xi) z(\eta) p(\eta)\, d\eta$$

This can be solved by successive approximations which gives for $k > 0$

$$z(\xi) = \frac{\sin k\xi}{k n(0)^{1/4}} + O\left(\frac{1}{k^2}\right) \qquad \text{and} \qquad z'(\xi) = \frac{\cos k\xi}{n(0)^{1/4}} + O\left(\frac{1}{k}\right)$$

To fix the idea let $n(1) \geq 1$ and let

$$\delta := \int_0^1 \sqrt{n(t)} dt$$

and going back to the original variable we obtain

$$d(k) = \frac{1}{k[n(0)n(1)]^{1/4}} \left(\sqrt{n(1)} \sin k \cos(k\delta) - \cos k \sin(k\delta)\right) + O\left(\frac{1}{k^2}\right)$$

$$= \frac{1}{k[n(0)n(1)]^{1/4}} \left(\sqrt{n(1)} \sin(k(1-\delta)) - (1 - \sqrt{n(1)}) \cos k \sin(k\delta)\right) + O\left(\frac{1}{k^2}\right)$$

TE for Spherically Symmetric Media

Theorem

Let $n \in C^2[0,1]$ be positive, and either $\int_0^1 \sqrt{n(t)}dt \neq 1$ or $n(1) \neq 1$. Then there exists an infinite number of real eigenvalues $k > 0$ accumulating at $+\infty$.

In the case when $n \neq 1$ is a positive constant then

$$kd(k) = \sin\sqrt{n}k\cos k - \sqrt{n}\cos\sqrt{n}k\sin k = 0$$

Examples

- When $n(r) = 1/4$ then $d(k) = -\frac{2}{3}\sin^3\left(\frac{k}{2}\right)$
  hence it has infinitely many real zeros and no complex zeros.
- When $n(r) = 4/9$ then $d(k) = -\frac{1}{k}\sin^3\left(\frac{k}{3}\right)\left[3 + 2\cos\left(\frac{2k}{3}\right)\right]$
  hence it has infinitely many real and infinitely many complex zeros

TE for Spherically Symmetric Media

Theorem (Colton-Leung (2015))

Let $n \neq 1$ be a positive constant. If $\sqrt{n}$ is an integer or reciprocal of an integer then no complex eigenvalues with radial eigenfunctions exits. Otherwise, there exists infinitely many complex eigenvalues.

For a proof see also Chapter 7 in

A. Kirsch (2022), *An introduction to the mathematical theory of inverse problems*, 3rd Edition Springer.

The transmission eigenvalue problem is non-selfadjoint

More on the transmission eigenvalue problem for spherical symmetric media see Prof. Gintides lecture.

Aktusun-Gintides-Papanicolaou (2011), Cakoni-Colton-Gintides (2011), Leung-Colton (2013, 2015), Colton-Leung-Meng (2015), Petkov-Vodev (2016) .....

Transmission Eigenvalue Problem in General

The transmission Eigenvalue Problem

Given (n, D), $n \in L^\infty(D)$, and let $n_* = \inf_{x \in D} n(x)$, $n^* = \sup_{x \in D} n(x)$.

Formulation 1

Find nonzero $w \in H_0^2(D)$ and $v \in L^2(D)$ satisfying

$$\Delta w + k^2 n w = k^2(1-n)v \qquad \text{in } D$$

$$\Delta v + k^2 v = 0 \qquad \text{in } D$$

$$w = \frac{\partial w}{\partial \nu} = 0 \qquad \text{on } \partial D$$

Formulation 2

Find nonzero $u \in L^2(D)$ and $v \in L^2(D)$ with $u - v \in H^2(D)$ satisfying

$$\Delta v + k^2 v = 0 \quad \text{and} \quad \Delta u + k^2 n u = 0 \qquad \text{in } D$$

$$u - v = 0 \quad \text{and} \quad \frac{\partial u}{\partial \nu} - \frac{\partial v}{\partial \nu} = 0 \qquad \text{on } \partial D$$

The Transmission Eigenvalue Problen

Exercise 1: Prove that Formulation 1 and Formulation 2 are equivalent.

There is a third formulation

Cakoni-Kress (2017), Vodev (2018), Ambrose-Cakoni-Moskow (2022.

Formulation 3

TE can be also viewed as values of $k \in \mathbb{C}$ for which the operator

$$\mathcal{N}_{k,n} - \mathcal{N}_{k,1} \qquad \text{has non-trivial kernel}$$

where the Dirichlet to Neumann operator

$$\mathcal{N}_{k,b} : H^{s+1/2}(\partial D) \to H^{s-1/2}(\partial D)\ ,\ s \in (-1,1) \text{ maps}$$

$$f \mapsto \frac{\partial \varphi}{\partial \nu} \quad \text{with } \varphi \text{ satisfying}$$

$$\Delta\varphi + k^2 b\varphi = 0 \quad \text{in } D \qquad \varphi = f \quad \text{on } \partial D$$

Exercise 2: Prove that Formulation 3 and Formulation 2 are equivalent.

The transmission Eigenvalue Problem ($*$)

Given D, $n = n_1 + \frac{i}{k} n_2$ $n \in L^\infty(D)$, $n_2 \geq 0$

Theorem

If $\Im(n_2(x)) > 0$ for x on some open set $A \subset D$ then all transmission eigenvalues are complex.

Proof: From Formulation 2 we have for $k > 0$ and $w \in H^2(\mathbb{R}^3)$

$$\Delta w + k^2 w = k^2(1-n)u \quad \text{in} \quad D \quad \text{and} \quad w = 0 \text{ in } \mathbb{R}^3 \setminus D$$

Multiplying by u the conjugate and integrate over D gives

$$\int_D u(\Delta \overline{w} + k^2 \overline{w})\, dx = k^2 \int_D (1-\overline{n})|u|^2\, dx \qquad \text{but}$$

$$\int_D u(\Delta \overline{w} + k^2 \overline{w})\, dx = \int_D u(\Delta \overline{w} + k^2 n \overline{w})\, dx + k^2 \int_D (1-n) u \overline{w} dx$$

$$k^2 \int_D (1-\overline{n})|u|^2\, dx = k^2 \int_D (1-n) u \overline{w} dx = k^2 \int_D (1-n)|u|^2 dx - k^2 \int_D (1-n) u \overline{v} dx$$

$$= k^2 \int_D (1-n)|u|^2 dx - k^2 \int_D (\Delta w + k^2 w)\overline{v} dx = k^2 \int_D (1-n)|u|^2 dx.$$

Taking the imaginary part gives

$$\int_D n_2 |u|^2 dx = 0 \Longrightarrow u = 0 \text{ on } A$$

Unique continuation gives $u = 0$ in D and hence $v = 0$.