The transmission Eigenvalue Problem

From now on we assume n real and let $n_* = \inf_{x\in D} n(x)$, $n^* = \sup_{x\in D} n(x)$.

Assumption

To fix the idea we assume $n_* > 1$. Similar results can be obtain for $0 < n_* \leq n^* < 1$.

Dividing by $n-1$ in formulation 1 and applying $(\Delta + k^2)$ on both sides we obtain: Find $w \in H_0^2(D)$ satisfying fourth order equation

$$(\Delta + k^2)\frac{1}{n-1}(\Delta + k^2 n)w = 0$$

or in the variational form

$$\int_D \frac{1}{n-1}(\Delta w + k^2 n w)(\Delta\varphi + k^2\varphi)\, dx = 0 \qquad \text{for all } \varphi \in H_0^2(D)$$

Transmission Eigenvalues $(*)$

Expand the above and define the bounded linear selfadjoint operators based on Riesz Representation Theorem

$$(Tw,\varphi)_{H^2} = \int_D \frac{1}{n-1}\Delta w\,\Delta\overline{\varphi}\,\mathrm{d}x \quad \text{coercive} \quad (Tw,\varphi)_{H^2(D)} \geq \frac{1}{n_*-1}\|w\|^2_{H^2}$$

$$(T_1w,\varphi)_{H^2} = -\int_D \frac{1}{n-1}\left(\Delta w\,\overline{\varphi} + w\,\Delta\overline{\varphi}\right)\mathrm{d}x - \int_D \nabla w\cdot\nabla\overline{\varphi}\,\mathrm{d}x$$

$$(T_2w,\varphi)_{H^2(D)} = \int_D \frac{n}{n-1} w\,\overline{\varphi}\,\mathrm{d}x \qquad \text{non-negative.}$$

Letting $k^2 := \tau$ we obtain the quadratic pencil eigenvalue problem

$$Tw - \tau T_1 w + \tau^2 T_2 w = 0, \qquad w \in H_0^2(D) \qquad \text{or}$$

$$w - \tau K_1 w + \tau^2 K_2 w = 0, \qquad w \in H_0^2(D)$$

with selfadjoint compact operators

$$K_1 = T^{-1/2}T_1T^{-1/2} \quad \text{and} \quad K_2 = T^{-1/2}T_2T^{-1/2}$$

Transmission Eigenvalues

The transmission eigenvalue problem can be transformed to the eigenvalue problem

$$(\mathbb{K} - \xi\mathbb{I})U = 0, \qquad U = \begin{pmatrix} w \\ \tau K_2^{1/2} w \end{pmatrix}, \qquad \xi := \frac{1}{\tau}$$

for the non-selfadjoint compact operator

$\mathbb{K}\colon H_0^2(D) \times H_0^2(D) \to H_0^2(D) \times H_0^2(D)$ given by

$$\mathbb{K} := \begin{pmatrix} K_1 & -K_2^{1/2} \\ K_2^{1/2} & 0 \end{pmatrix}.$$

However from here one can see that the transmission eigenvalues form a discrete set with $+\infty$ as the only possible accumulation point.

Transmission Eigenvalues

To obtain existence of transmission eigenvalues, Faber-Krahn type inequalities, and monotonicity properties for real eigenvalues $\tau > 0$, we rewrite the transmission eigenvalue problem in the form

$$(\mathbb{A}_\tau - \tau\mathbb{B})w = 0 \quad \text{in} \;\; H_0^2(D)$$

$$(\mathbb{A}_\tau w, \varphi)_{H^2(D)} = \int_D \frac{1}{n-1}(\Delta w + \tau w)(\Delta\varphi + \tau\varphi)\,dx + \tau^2 \int_D w \cdot \varphi\,dx$$

$$(\mathbb{B}w, \varphi)_{H^2(D)} = \int_D \nabla w \cdot \nabla\varphi\,dx$$

Observe that

- The mapping $\tau \to \mathbb{A}_\tau$ is continuous from $(0, +\infty)$ to the set of self-adjoint coercive operators from $H_0^2(D) \to H_0^2(D)$.
- $\mathbb{B} : H_0^2(D) \to H_0^2(D)$ is self-adjoint, compact and non-negative.

Coercivity of $\mathbb{A}_\tau$

Let $\gamma = \frac{1}{n^*-1}$

$$
\begin{aligned}
(\mathbb{A}_\tau w, w)_{H^2(D)} &\geq \gamma\|\Delta w + \tau w\|^2_{L^2} + \tau^2\|w\|^2_{L^2} \\
&\geq \gamma\|\Delta w\|^2_{L^2} - 2\gamma\tau\|\Delta w\|_{L^2}\|w\|_{L^2} + (\gamma+1)\tau^2\|w\|^2_{L^2} \\
&= \epsilon\left(\tau\|w\|_{L^2} - \frac{\gamma}{\epsilon}\|\Delta w\|_{L^2(D)}\right)^2 + \left(\gamma - \frac{\gamma^2}{\epsilon}\right)\|\Delta w\|^2_{L^2(D)} \\
&+ (1+\gamma-\epsilon)\tau^2\|w\|^2_{L^2} \\
&\geq \left(\gamma - \frac{\gamma^2}{\epsilon}\right)\|\Delta w\|^2_{L^2(D)} + (1+\gamma-\epsilon)\tau^2\|w\|^2_{L^2}
\end{aligned}
$$

for some $\gamma < \epsilon < \gamma + 1$. Furthermore, since $\nabla w \in H^1_0(D)^2$, using the Poincaré inequality we have that

$$\|\nabla w\|^2_{L^2(D)} \leq \frac{1}{\lambda_1(D)}\|\Delta w\|^2_{L^2(D)}.$$

Hence we can conclude $\quad (\mathbb{A}_\tau w, w)_{H^2(D)} \geq C_\tau\|w\|^2_{H^2(D)} \quad$ for some constant $C_\tau > 0$

As for $\mathbb{B}$: The compact embedding of $H^2(D)$ into $H^1(D)$ and the fact that $\nabla w \in H^1_0(D)$ imply that $\mathbb{B} : H^2_0(D) \to H^2_0(D)$ is compact since $\|\mathbb{B}w\|_{H^2(D)} \leq c\|w\|_{H^1(D)}$.

Transmission Eigenvalues

Now we consider the generalized eigenvalue problem

$$(\mathbb{A}_\tau - \lambda(\tau)\mathbb{B})u = 0 \quad \text{in} \;\; H_0^2(D)$$

Note that $k^2 = \tau$ is a transmission eigenvalue if and only if satisfies $\lambda(\tau) = \tau$

For a fixed $\tau > 0$ there exists an increasing sequence of eigenvalues $\lambda_j(\tau)_{j\geq 1}$ such that $\lambda_j(\tau) \to +\infty$ as $j \to \infty$.

These eigenvalues satisfy

$$\lambda_j(\tau) = \min_{W\subset\mathcal{U}_j}\left(\max_{u\in W\setminus\{0\}} \frac{(\mathbb{A}_\tau u, u)}{(\mathbb{B}u, u)}\right).$$

$\mathcal{U}_j$ denotes the set of all j-dimensional subspaces W of $H_0^2(D)$ such that $W \cup \text{Kern}(\mathbb{B}) = \{0\}$

Transmission Eigenvalues

Hence, if there exists two positive constants $\tau_0 > 0$ and $\tau_1 > 0$ such that both Assumptions holds

1. $\mathbb{A}_{\tau_0} - \tau_0 \mathbb{B}$ is positive on $H_0^2(D)$
2. $\mathbb{A}_{\tau_1} - \tau_1 \mathbb{B}$ is non positive on a k dimensional subspace of $H_0^2(D)$

then each of the equations $\lambda_j(\tau) = \tau$ for $j = 1, \dots, k$, has at least one solution in $[\tau_0, \tau_1]$ meaning that there exists k transmission eigenvalues (counting multiplicity) within the interval $[\tau_0, \tau_1]$.

See e.g. Section 4.1 in

F. Cakoni, D. Colton and H. Haddar (2023), *Inverse Scattering Theory and Transmission Eigenvalues*, CBMS-NSF, SIAM Publication, 2nd Edition.

It is now obvious that determining such constants τ_0 and τ_1 provides the existence of transmission eigenvalues as well as the desired bounds on the eigenvalues.

Transmission Eigenvalues

Obviously we have

$$\left((\mathbb{A}_\tau - \tau\mathbb{B})w, w\right) := \int_D \frac{1}{n-1} \left|(\Delta w + \tau n w)\right|^2 \, dx + \tau \int_D \left(|\nabla w|^2 - \tau n |w|^2\right) dx.$$

Poincare inequality yields the Faber-Krahn type inequality for the first real transmission eigenvalue (not isoperimetric)

$$\tau_1 > \frac{\lambda_1(D)}{\sup_D n}.$$

where $\lambda_1(D)$ is the first Dirichlet eigenvalue of $-\Delta$ in D.

1. In particular all $\tau_0 \in (0, \lambda_1(D)/\sup_D n)$ satisfies Assumption 1.
2. Concerning Assumption 2

Transmission Eigenvalues

Take τ_1 be the square of the first real eigenvalue of a ball $B_r \subset D$ of radius r and constant refractive index n_*.

Let u_r be the corresponding eigenfunction and denote $\tilde{u}_r \in H_0^2(D)$ its extension by zero to the whole of D. Then

$$(\mathbb{A}_{\tau_1}\tilde{u}_r - \tau_1 \mathbb{B}\tilde{u}_r, \tilde{u}_r) \leq 0.$$

If r is such that $m(r)$ disjoint balls can be included in D, the above condition is satisfied in a $m(r)$-dimensional subspace of $H_0^2(D)$

There exists $m(r)$ transmission eigenvalues (counting multiplicity). As $r \to 0$, $m(r) \to \infty$ and since the multiplicity of an eigenvalue is finite we prove the existence of an infinite set of real transmission eigenvalues

Monotonicity Properties

$n^* := \sup_D n$ and $n_* := \inf_D n$, $\tau := k^2$

Theorem (Cakoni-Gintides-Haddar (2010))

If $n \in L^\infty(D)$ and $n_* > 1$, then there exists a sequence of real eigenvalues $k_j(n, D)$ accumulating to $+\infty$. Futhermore, and they satisfy monotonicity properties:

$$k_j(n^*, B_1) \leq k_j(n^*, D) \leq k_j(n(x), D) \leq k_j(n_*, D) \leq k_j(n_*, B_2)$$

where $B_2 \subset D \subset B_1$ are balls.

Proof: Existence is proven. For simplicity we show the proof of monotonicity for k_1

$$\lambda_j(\tau, D, n(x)) = \min_{W \in \mathcal{U}_j} \max_{\substack{w \in W \\ \|\nabla w\|_{L^2} = 1}} \int_D \frac{1}{n(x) - 1} |\Delta w + \tau w|^2 \, dx + \tau^2 \|w\|^2_{L^2(D)}$$

$\forall\ u \in H_0^2(D)$ such that $\|\nabla u\|_{L^2(D)} = 1$ we have

$$\frac{1}{n^* - 1} \|\Delta u + \tau w\|^2_{L^2(D)} + \tau^2 \|w\|^2_{L^2(D)} \quad \leq \quad \int_D \frac{1}{n(x) - 1} |\Delta w + \tau w|^2 \, dx + \tau^2 \|w\|^2_{L^2(D)}$$

$$\leq \frac{1}{n_* - 1} \|\Delta w + \tau w\|^2_{L^2(D)} + \tau^2 \|w\|^2_{L^2(D)}.$$

Proof Cont.

Therefore we have that for an arbitrary $\tau > 0$

$$\lambda_1(\tau, B_2, n^*) \leq \lambda_1(\tau, D, n^*) \leq \lambda_1(\tau, D, n(x)) \leq \lambda_1(\tau, D, n_*) \leq \lambda_1(\tau, B_1, n_*).$$

- For $\tau_1 := k_1^2(n_*, B_1)$, $B_1 \subset D$, from the proof of existence we have that

$$\lambda_1(\tau_1, D, n(x)) - \tau_1 \leq 0$$

- On the other hand, for $\tau_0 := k_1^2(n^*, B_2)$, $D \subset B_2$, we have

$$\lambda_1(\tau_0, B_2, n^*) - \tau_0 = 0$$

  and hence

$$\lambda_1(\tau_0, D, n(x)) - \tau_0 \geq 0$$

- Therefore the first eigenvalue $k_1(D, n(x))$ is between $k_1(n^*, B_2)$ and $k_1(n_*, B_1)$.

Inequalities

1. If $1 < \alpha \le n_1(x) \le n_2(x)$ for almost all $x \in D$, then
$$k_j(n_2(x), D) \le k_j(n_1(x), D)$$
2. If $0 < \alpha \le n_1(x) \le n_2(x) \le \beta < 1$ for almost all $x \in D$, then
$$k_j(n_1(x), D) \le k_j(n_2(x), D).$$

Exercise: Modify the above argument to prove part 1.

Theorem (Theorem 4.18, Cakoni-Colton-Haddar CBMS (2023))

For known D, the constant n is uniquely determined from the corresponding smallest transmission eigenvalue $k_1(n, D) > 0$ provided that it is known a priori that either $n > 1$ or $0 < n < 1$.

Sing Changing $n-1$

Let $\mathcal{N}$ be a neighborhood of ∂D form inside D, and let

$$n_\star = \inf_{x\in\mathcal{N}} n(x) \qquad \text{and} \qquad n^\star = \sup_{x\in\mathcal{N}} n(x)$$

Theorem

Assume that $n \in L^\infty(D)$ with $n(x) > n_0 > 0$ for almost all $x \in D$ and either $n^\star < 1$ or $n_\star > 1$ for some neighborhood $\mathcal{N}$ of the boundary ∂D. Then the set of transmission eigenvalues is at most discrete with $+\infty$ as the only accumulation point.

Sylvester, SIAM J. Math. Anal., 44 (2012), Kirsch, Math. Acad. Sci. Paris, 354 (2016).

Sketch of the Proof: We work on the space $(w, v) \in X(D) = H_0^2(D) \times L^2(D)$. Find $(w, v) \in X(D)$ suwch that for all $(\psi, \varphi) \in X(D)$

$$\int_D (\Delta\overline{\psi} + k^2\overline{\psi})v\,dx + \int_D (\Delta w + k^2 nw)\overline{\varphi} + (n-1)v\overline{\varphi}\,dx$$

Denote

$$\mathcal{A}_k(w, v; \psi, \varphi) := \int_D (\Delta\psi + k^2\psi)v\,dx + \int_D (\Delta w + k^2 nw)\varphi + (n-1)v\varphi\,dx$$

$$\hat{\mathcal{A}}_k(w, v; \psi, \varphi) := \int_D (\Delta\overline{\psi} + k^2\overline{\psi})v\,dx + \int_D (\Delta w + k^2 w)\overline{\varphi} + (n-1)v\overline{\varphi}\,dx$$

and the corresponding operators by means of Riesz representation theorem

$$\mathcal{A}_k(w, v; \psi, \varphi) = \langle A_k(w, v), (\psi, \varphi)\rangle_{X(D)} \qquad \hat{\mathcal{A}}_k(w, v; \psi, \varphi) = \left\langle \hat{A}_k(w, v), (\psi, \varphi)\right\rangle_{X(D)}$$

Sing Changing $n-1$ - Proof cont,

Thus we need to study the kernel $A_k(w,v)=0$.

Step 1: For any two $k_1, k_2 \in \mathbb{C}$, $A_{k_1} - \hat{A}_{k_2}$ and $A_{k_1} - A_{k_2}$ are compact.

Note $\left(\mathcal{A}_{k_1} - \hat{\mathcal{A}}_{k_2}\right)(w_j, v_j; \psi, \varphi) = (k_1^2 - k_2^2)\int_D \overline{\psi} v_j \, dx + \int_D (k_1^2 n - k_2^2)\, w_j \overline{\varphi}\, dx.$

Step 2: There exists a $\kappa_0 > 0$ and a constant $c > 0$ s. th. for all $\kappa \geq \kappa_0$

$$\sup_{(\psi,\varphi)\neq 0} \frac{\left|\hat{\mathcal{A}}_{i\kappa}(w, v; \psi, \varphi)\right|}{\|(\psi, \varphi)\|_{X(D)}} \geq c\|(w,v)\|_{X(D)} \qquad \text{for all } (w,v) \in X(D).$$

In particular the operator $\hat{A}_{i\kappa}$ is invertible, A_k is Fredholm of index zero.

(Analytic Fredholm Theory)

Step 3: For some $\kappa > 0$, the operator $A_{i\kappa} : X(D) \to X(D)$ is invertible with bounded inverse. Thus the kernel of A_k is non-zero for at most a discrete set of $k \in \mathbb{C}$ with $+\infty$ as possible accumulation point.

Sing Changing $n-1$ - Proof cont,

Lemma (An important estimate)

There exist constants $c>0$ and $d>0$ such that for all $k=i\kappa$, $\kappa>0$, the following estimate holds:

$$\int_{D\setminus\mathcal{N}}|v|^2\,dx \le ce^{-2d\kappa}\int_{\mathcal{N}}|n-1||v|^2\,dx$$

for all solutions $v\in L^2(D)$ of $\Delta v-\kappa^2 v=0$ in D.

Will show this when $n^\star<1$. It suffices to prove that $A_{i\kappa}:X(D)\to X(D)$ is injective for some κ since $\hat{A}_k:X(D)\to X(D)$ is Fredholm of index zero.

By contradiction: we assume $\forall\ \kappa>0$ there are $(w,v)\in X(D)$ with $\|(w,v)\|_{X(D)}=1$ and $A_{i\kappa}(w,v)=0$.

In terms of PDEs this means $w\in H_0^2(D)$ and $v\in L^2(D)$ satisfy in D

$$\Delta w-\kappa^2 nw=(1-n)v \qquad (1)$$

$$\Delta v-\kappa^2 v=0$$

Sing Changing $n-1$ - Proof cont,

Multiplying (1) by $\overline{v}$, integrating over D and using Green's second identity and the second equation in (1) yields

$$\int_D \kappa^2(n-1)w\overline{v}\,dx = \int_D (n-1)|v|^2\,dx.$$

Multiplying (1) by $\overline{w}$, integrating over D yields

$$\int_D \left[|\nabla w|^2 + \kappa^2 n|w|^2\right]\,dx = \int_D (n-1)v\overline{w}dx = \frac{1}{\kappa^2}\int_D (n-1)|v|^2\,dx.$$

Since $n>0$ in D we see that $\int_D (n-1)|v|^2\,dx > 0$.

Recalling that $n^\star := \sup_{\mathcal{N}} n < 1$ from Lemma it follows

$$\begin{aligned}\int_D (n-1)|v|^2\,dx &= \int_{\mathcal{N}} (n-1)|v|^2\,dx + \int_{D\setminus\mathcal{N}} (n-1)|v|^2\,dx \\ &= \left(1 + ce^{-2d\kappa}\|n-1\|_{L^\infty(D)}\right)\int_{\mathcal{N}} (n-1)|v|^2\,dx < 0\end{aligned}$$

for $\kappa > 0$ sufficiently large, which is a contradiction.

The State of the Art of TE problem

- Discreteness and existence of real TE: $n-1$ is one sign in $D \setminus D_0$, $\overline{D}_0 \subset D$ where $n \equiv 1$, or $n(x) - 1 = c\rho(x)^\beta$ $\beta > -1$ and $\rho(x) = \inf_{y \in \partial D} |x - y|$. Cakoni-Gintides-Haddar (2010), Serov (2014)
- If $n \in L^\infty(D)$, ∂D and Lipschitz and $n \neq 1$ is one sign in a neighborhood of ∂D, transmission eigenvalues are discrete with ∞ as the only possible accumulation point Sylvester (2012), Kirsch (2014)
- If $n \in C^1$ near ∂D in C^2 and $n \neq 1$ on ∂D completeness of generalized eigenfunctions and Weyl's law for counting function are proven. Robianno (2013), H.M. Nguyen-J. Fornerod (2022)
- If $n \in C^\infty(\overline{D})$, ∂D is C^∞ and $n \neq 1$ on ∂D, transmission eigenvalues lie in a strip around the real axis Vodev (2018)

Open Problem: Is $n-1$ one sign in a neighborhood of ∂D necessary for discreteness?

Ambrose-Cakoni-Moskow, RMS (2022), Kian-Uhlmann (2023) arXiv:2302.03457

Determination of Real Transmission Eigenvalues

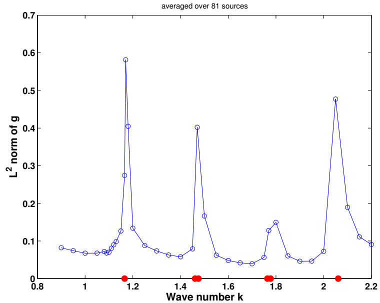
(Based on Linear Sampling Method)

Assume k is not a non-scattering number. Let $g_z^\alpha \in L^2(\mathbb{S})$ be the regularized solution of $F_k g = \Phi^\infty(\cdot, z)$

$$(\alpha I + F_k^* F_k) g_z^\alpha = F^* \Phi^\infty(\cdot, z), \qquad \Phi(\cdot,\cdot) \text{ fundamental solution of HE.}$$

For any ball $B \subset D$, $\|H g_z^\alpha\|_{L^2(D)}$ is bounded for $z \in B$ as $\alpha \to 0$ if and only if $k > 0$ is not a transmission eigenvalue.

Cakoni-Colton-Haddar (2010), Cakoni-Colton-Haddar Chapter 4 CBMS (2023)



Underlying Idea of The Proof

Assume D is known $n-1$ one sign in D. Recall that $F = H^* TH$.

Case 1 k is not a transmission eigenvalue

Coercive and continuous of T gives $\beta \|Hg\|^2 \leq |(Fg, g)| \leq \|T\| \|Hg\|^2$ for some $\beta > 0$.

From factorization method $\phi_z := \Phi_\infty(\cdot, z)$ is in the range of $(F_k^* F_k)^{1/4}$ for $z \in D$. Using the eigensystem $(\lambda_j, \psi_j)_{j \geq 1}$ of the normal operator F_k, we observe that

$$g_z^\alpha = \sum_j \frac{\overline{\lambda_j}}{\alpha + |\lambda_j|^2} (\phi_z, \psi_j) \psi_j,$$

Therefore $|(F_k g_z^\alpha, g_z^\alpha)| = \left| \sum_j \frac{|\lambda_j|^2 \overline{\lambda_j}}{(\alpha + |\lambda_j|^2)^2} |(\phi_z, \psi_j)|^2 \right| \leq \sum_j \frac{|\lambda_j|^3}{(\alpha + |\lambda_j|^2)^2} |(\phi_z, \psi_j)|^2.$

On the other hand by coercivity of S in $F = |F|^{1/2} S |F|^{1/2}$ $\quad |(F g_z^\alpha, g_z^\alpha)| \geq \beta \sum_j \frac{|\lambda_j|^3}{(\alpha + |\lambda_j|^2)^2} |(\phi_z, \psi_j)|^2$

Recall $|F|^{1/2} \psi = \sum \sqrt{|\lambda_j|} (\psi, \psi_j) \psi_j$. Since ϕ is in the range of $(F_k^* F_k)^{1/4}$, the Picard criterion gives

$$\sum_j \frac{1}{|\lambda_j|} |(\phi_z, \psi_j)|^2 < +\infty.$$

Consequently, since $\frac{|\lambda_j|^3}{(\alpha + |\lambda_j|^2)^2} \to \frac{1}{|\lambda_j|}$ as $\alpha \to 0$ and $\frac{|\lambda_j|^3}{(\alpha + |\lambda_j|^2)^2} \leq \frac{1}{|\lambda_j|}$,

we get that $\limsup_{\alpha \to 0} |(F g_z^\alpha, g_z^\alpha)| < +\infty$ thus $\|H g_z^\alpha\|_{L^2(D)} < +\infty$

Underlying Idea of The Proof-cont.

Case 2 k is a transmission eigenvalue

Since F_k has dense range $Fg_z^\alpha \to \Phi^\infty(\cdot\,,z)$.

Assume that there is a ball $B \subset D$ such that for a.e. $z \in B$, $\|Hg_z^\alpha\|_{L^2(D)} \leq M$ as $\alpha \to 0$ (the constant M may depend on z).

Then for fixed z there exists a subsequence $v_n = Hg_z^{\alpha_n}$ that weakly converges to v_z a L-solution of the Helmholtz equation.

We know $F_k = GH$. Since G is a compact operator, we deduce that $Gv_z = \Phi^\infty(\cdot\,,z)$. Rellich's lemma gives

$$\Delta w_z + k^2 n w_z = k^2(1-n)v_z \qquad \text{and} \qquad \Delta v_z + k^2 v_z = 0 \qquad \text{in} \;\; D$$

$$w_z = \Phi^\infty(\cdot\,,z) \qquad \text{and} \qquad \frac{\partial w_z}{\partial \nu} = \frac{\partial \Phi^\infty(\cdot\,,z)}{\partial \nu} \qquad \text{on} \;\; \partial D$$

We write the equation $\displaystyle\int_D \frac{1}{n-1}(\Delta w_z + k^2 w_z)(\Delta\varphi + k^2 n\varphi)dx = 0 \quad \forall \varphi \in H_0^2(D)$

Since k is a transmission eigenvalue, there exists a non $w_0 \in H_0^2(D)$ satisfying

$$(\Delta + k^2)\frac{1}{n-1}(\Delta w_0 + k^2 n w_0) = 0 \text{ in } D.$$

Taking $\varphi = w_0$ and applying Green's theorem twice yields,

$$\int_{\partial D}\left(\frac{1}{n-1}(\Delta w_0 + k^2 n w_0)\right)\;\frac{\partial \Phi(\cdot\,,z)}{\partial \nu}\;ds - \int_{\partial D}\frac{\partial}{\partial \nu}\left(\frac{1}{n-1}(\Delta w_0 + k^2 n w_0)\right)\;\Phi(\cdot\,,z)\;ds = 0,$$

(Integrals have to be understood in the sense of $H^{\mp 1/2}(\partial D)$ (resp. $H^{\mp 3/2}(\partial D)$) duality pairing.

Underlying Idea of The Proof-cont

Defining $\Psi(x) := \frac{1}{n(x)-1}(\Delta + k^2 n(x))w_0(x)$ in D, we observe that

$$\Delta\psi + k^2\psi = 0 \quad \text{in } D.$$

Classical interior elliptic regularity results and the Green's representation theorem imply that

$$\Psi(z) = \int_{\partial D} \left(\Psi(x)\, \frac{\partial \Phi(x,z)}{\partial \nu} - \frac{\partial \Psi(x)}{\partial \nu} \Phi(x,z) \right) ds_x \qquad \text{for } z \in D.$$

Above and the unique continuation principle now show that $\Psi = 0$ in D.

Therefore

$$(\Delta + k^2 n(x))w_0(x) = 0 \;\text{ in }\; D.$$

Since $w_0 \in H_0^2(D)$ one deduces again from the unique continuation principle that $w_0 = 0$ in D, which is a contradiction.

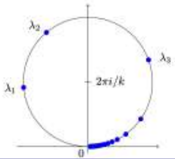
Determination of Real Transmission Eigenvalues

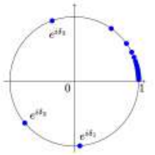
Assume $n > 1$ in D.

Lechleiter-Kirsch (2013)

Audibert-Chenel-Haddar (2018)

Cakoni-Colton-Haddar (2023)





(Based on Inside Outside Duality)

Let $(\lambda_j(k), g_j(k)$ be the eigensystem of the normal operator F_k. Denote

$$\lambda_j(k) = \tfrac{2\pi}{ik}\left(e^{i\delta_j(k)} - 1\right) \text{ and } \sigma_1(k) \text{ be the largest phase.}$$

- If $k > 0$ is not a transmission eigenvalue $\delta_j(k) \to 0$.
- If there is a sequence $\{k_\ell\} \to k_0 > 0$ such that $\delta_1(k_\ell) \to 2\pi$ as $\ell \to \infty$, then k_0 is a transmission eigenvalue.
- Let v be v-part of eigenfunction for k_0. Then

$$v_\ell := \frac{v_{g_z^\alpha}(k_\ell)}{\|v_{g_z^\alpha}(k_\ell)\|_{L^2(D)}} \to v \text{ as } \ell \to \infty$$

Application of Transmission Eigenvalues

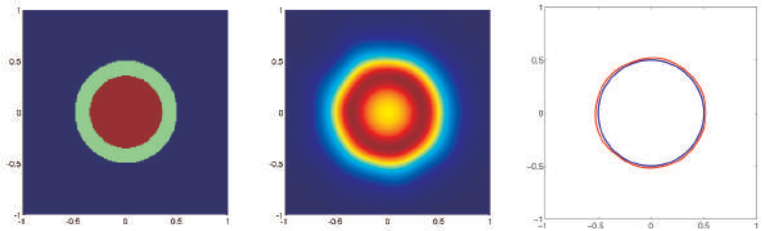
Let k_1 be the first transmission eigenvalue and suppose $n(x) > 1$ for $x \in \overline{D}$. Then, given k_1 and a knowledge of D, a constant n_0 can be determined such that the scattering problem for $n(x) = n_0$ also has k_1 as its first transmission eigenvalue. Then

$$\min_{\overline{D}} n(x) \leq n_0 \leq \max_{\overline{D}} n(x).$$

$$n_0 \approx \frac{1}{|D|} \int_D n(x) dx$$

- Flaws or voids in D can be detected by changes in n_0.
- Higher eigenvalues may be used for more information (see Drossos Gintides talk).

Numerical Example: Inhomogeneous Isotropic Media



n_e	n_i	k_1	n_0-exact shape	n_0-recon. shape
8	8	2.98	8.07	7.61
11	5	3.27	7.05	6.69
22	19	1.76	20.28	18.86
67	61	0.97	64.11	59.42

Non-scattering Wave Numbers