TE and Non-Scattering

$k > 0$ is a transmission eigenvalue

if there are nonzero v and $u \in H_0^2(D)$ such that

$$\Delta w + k^2 n w = k^2(1-n)v \qquad \text{in } D$$

$$w = 0 \qquad \text{and} \qquad \frac{\partial w}{\partial \nu} = 0 \qquad \text{on } \partial D$$

$$\text{and} \qquad \Delta v + k^2 v = 0 \qquad \text{in } D$$

$k > 0$ is a non-scattering wave number

if there a solution $w \in H_0^2(D)$ of this problem

$$\Delta w + k^2 n w = k^2(1-n)v_g \qquad \text{in } D$$

$$w = 0 \qquad \text{and} \qquad \frac{\partial w}{\partial \nu} = 0 \qquad \text{on } \partial D \qquad \text{with}$$

$$v_g(x) = \int_{\mathbb{S}} e^{ikx\cdot\hat{y}} g(\hat{y})\, ds_{\hat{y}} \qquad (\Delta v_g + k^2 v_g = 0 \quad \text{in } \mathbb{R}^3)$$

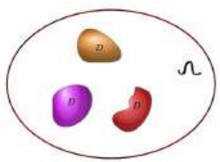
Non-scattering Configuration

k is a non-scattering wave number, if there exists a solution to

$$\Delta w + k^2 n w = k^2(1 - n)v \qquad \text{in } D$$

$$w = 0 \quad \text{and} \quad \nu \cdot \nabla w = 0 \quad \text{on } \partial D$$

$$\text{with } v \text{ satisfying} \quad \Delta v + k^2 v = 0 \qquad \text{in } \Omega$$



Note that v real analytic in a region $\Omega \supset \overline{D}$.

If $k > 0$ is a transmission eigenvalue, can the part v of the eigenfunction be extended as solution to the Helmholtz equation outside D?

In other words is the part v of the eigenfunction sufficiently regular up to the boundary of D?

Non-Existence of Non-scattering Wave Numbers

If D contains a boundary point $x_0 \in \partial D$ that is a corner in $\mathbb{R}^2$, or a vertex, conical corner, edge point in $\mathbb{R}^3$, and $n(x_0) \neq 1$ and $n \in C^{1,\alpha}$ locally in $B_\epsilon(x_0)$, then every incident wave is scattered by D, n.

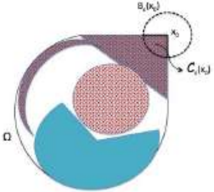
Letting $\mathcal{C}_\epsilon := B_\epsilon \cup D$. There are no non-trial u and v such that

$$\Delta u + k^2 n u = 0 \qquad \text{in } \mathcal{C}_\epsilon$$

$$\Delta v + k^2 v = 0 \qquad \text{in } B_\epsilon$$

$$u - v = 0 \qquad \text{on } \partial D \cap B_\epsilon$$

$$\frac{\partial u}{\partial \nu} - \frac{\partial v}{\partial \nu} = 0 \qquad \text{on } \partial D \cap B_\epsilon$$



No assumption on the incident field v is needed!

This result was first proven by Blåsten-Päivärinta-Sylvester (2013)

Hu-Salo-Vesalainen (2016), Elschner-Hu (2017), (2018)

Blåsten (2018), Cakoni-Xiao (2019), Blåsten-Liu (2020)

Two Techniques for Corner Scattering

- Based on CGO (rapidly decaying) solutions of the Helmholtz equation.

  Blåsten-Päivärinta-Sylvester (2013), Päivärinta-Sylvester-Vesalainen (2017), Blåsten (2018), Cakoni-Xiao (2019), Xiao (2021)

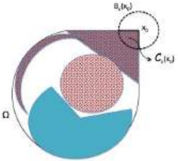
  CGO solution is used as test function w in

$$\int_{\mathcal{C}_\epsilon} (1-n) v \varphi \, dx = \int_{\mathcal{K}_\epsilon} \varphi \frac{\partial u}{\partial \nu} - u \frac{\partial \varphi}{\partial \nu} \, ds$$

  to control the boundary terms, where u and v are transmission eigenfunctions.

- Based on singularity analysis of the transmission eigenfunctions in a neighborhood of the boundary singularity.

  Elschner-Hu (2017), (2018)



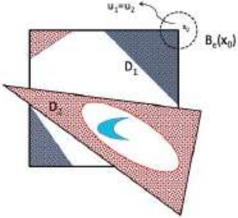
In both methods a contradiction is achieved if v is assumed to solve the Helmholtz equation in $B_\epsilon(x_0)$.

Uniqueness of Polyhedron with One Measurement

This negative result implies that scattering data due to one single incident plane wave uniquely determines the support of convex polyhedron inhomogeneities. Assumption is that $n \in C^{1,\alpha}$ near ∂D and $n \neq 1$ on ∂D.

Hu-Salo-Vesalainen (2016), Elschner-Hu (2018), Blåsten (2018), Cakoni-Xiao (2019), Blåsten-Liu (2021)

Proof in $\mathbb{R}^2$: Assume there are convex polyhedron D_1 and D_2 that such that $u_1^\infty = u_2^\infty$ due to one incident plane wave $u^i = e^{ikx\cdot\hat{y}}$ (or point source or any single experiment). By Rellich's Lemma the total field $u_1 = u_2$ up to the boundary of $\mathbb{R}^2 \setminus (D_1 \cup D_2)$. Let x_0 be the vertex of a corner of D_1 outside D_2. Then in a sufficiently small ball we have that the set of equations holds with $u := u_1$ and $v := v_2$ and $D := D_1$, which is a contradiction.



Singularities Scatter

For general domains D this question is only recently studied.
Partial results: Blåsten-Liu (2021), Vogelius-Xiao (2021)

Major progress using free boundary methods in:

F. Cakoni and . Vogelius (2021), Singularities almost always scatter: Regularity results for non-scattering inhomogeneities, *Communications in Pure and Applied Math* (to appear).

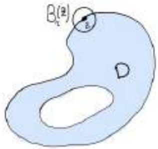
M. Salo and H. Shahgholian (2021), Free boundary methods and non-scattering phenomena, *Research in the Mathematical Sciences*.

Almost All Singularities Scatter

Let ∂D be Lipschitz, $n \in L^\infty(D)$. The nontrivial incident field v is scattered if there is $z \in \partial D$ such that the following cannot hold

$$\Delta w + k^2 n w = k^2(1-n)\Re(v) \quad \text{in } D \cap B_r(z)$$

$$w = \frac{\partial w}{\partial \nu} = 0 \quad \text{on } \partial D \cap B_r(z)$$



Theorem (Cakoni-Vogelius 2021)

Incident field v scatterers if $\exists z \in \partial D$ where $(n(z)-1)v(z) \neq 0$ and

- n in $C^{\ell,\mu}(\overline{D} \cap B_r(z))$, $\ell \geq 1$, and $\partial D \cup B_\rho(z)$ is not in $C^{\ell+1,\mu}$ $\forall\ \rho$.
- n in C^∞ in $\overline{D} \cap B_r(z)$ and $\partial D \cap B_\rho(z)$ is not C^∞ $\forall\ \rho$.
- n is real analytic at z and $\partial D \cap B_\rho(z)$ is not analytic $\forall\ \rho$.

Incident field v is real analytic as solution of $\Delta v + k^2 v = 0$ in $\Omega \supset \overline{D}$, but only regularity of v up to ∂D matters.

The Idea of the Proof

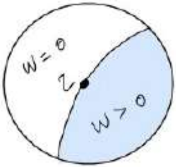
- Higher regularity straightforward application of the celebrated paper by Kinderlehrer and Niremberg (1977) provided ∂D is C^1, w is C^2 near z.
- To obtain this regularity from Lipschitz, we appeal to free boundary methods due to Caffarelli (1977) which apply to problems

$$\Delta w = f\chi_{\{w\neq 0\}} \qquad \text{in } B_r(z)$$

$$z \in \partial\{w = |\nabla w| = 0\}$$

f is Lipshitz up to the boundary and $w > 0$.

For us $f := -k^2 w + k^2(1-n)\Re(v)$



Most of the work is to prove

- $w \in C^{1,1}$ up to the boundary. Default regularity of w is $C^{1,\alpha}$ $\alpha < 1$. We use that w is zero outside D to improve it.
- We then use $f \in C^{1,1}(\overline{D} \cap B_r(z))$ and the non-degeneracy condition $[(1-n)\Re(v)]\,(z) \neq 0$ to prove one sign condition on w

Remarks

- We need the non-vanishing condition $v(z) \neq 0$ on incident waves at the boundary singularity.

  If k^2 is not a Dirichlet eigenvalue of $-\Delta$ in D there are many Herglotz wave function that do not vanish on the boundary.

  Question: Can a $k > 0$ be transmission eigenvalue for n, D and k^2 a Dirichlet eigenvalue of $-\Delta$ in D?

- Salo-Shangholian (2021) remove Lipschitz starting regularity. One can start merely with a solid region ($\text{int}\overline{D} = D$), but allowing for the possibility that D has inward cusps (is thin at boundary points).

- This result provides lack of sufficient regularity of v part of the eigenfunction near a boundary singularity or otherwise vanishing.

- Our result establishes necessary condition for an inhomogeneity to be non-scattering. For general smooth inhomogeneity (other than balls) the existence of non-scattering wave numbers is still open.

Connection to Schiffer's Property

Always Scatterers: Given v satisfying $\Delta v + k^2 v = 0$ in $\mathbb{R}^d$, the problem

$$\Delta u + k^2 n u = k^2(1-n)v \quad \text{in } D$$
$$u = 0, \ \frac{\partial u}{\partial \nu} = 0 \qquad \text{on } \partial D$$

has no solution for any $k > 0$.

D has Schiffer's property if the problem

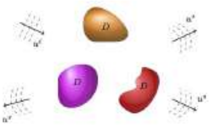
$$\Delta w + \lambda w = -1 \quad \text{in } D \qquad w = 0, \ \frac{\partial w}{\partial \nu} = 0 \quad \text{on } \partial D$$

has no solution for any λ.
Conjecture: The only simply connected domain in $\mathbb{R}^d$ that fails to have Schiffer's property are balls.

Integral geometric formulation of Schiffer's property is Pompeiu property.

A Glimpse on Anisotropic Media

Scattering by an Inhomogeneous Media



∂D is Lipschitz, $k = \omega / c_b$, $\rho_b = 1$

$n \in L^\infty(\mathbb{R}^d)$ real valued positive

$A \in L^\infty(\mathbb{R}^d)$ symmetric positive definite matrix

$$\mathrm{Supp}(A - I) \cup \mathrm{Supp}(n - 1) \qquad \text{is bounded}$$

- The incident field v satisfies the Helmholtz equation

$$\Delta v + k^2 v = 0 \qquad \text{in } \mathbb{R}^d$$

- The total field $u = w + v$ satisfies

$$\nabla \cdot A \nabla u + k^2 n u = 0 \qquad \text{in } \mathbb{R}^d$$

- The scattered field w is outgoing, i.e. it satisfies the Sommerfeld radiation condition

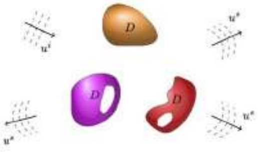
Scattering by an Inhomogeneous Media

The scattered field w satisfies

$$\nabla \cdot A \nabla w + k^2 n w = \nabla \cdot (I - A) \nabla v + k^2 (1 - n) v_g \qquad \text{in } \mathbb{R}^d$$

$$v_g(x) = \int_{\mathbb{S}} e^{ikx\cdot\hat{y}} g(\hat{y})\, ds_{\hat{y}} \qquad (\Delta v_g + k^2 v_g = 0 \quad \text{in } \mathbb{R}^d)$$

w satisfies the (outgoing) Sommerfeld radiation condition.



$$\overline{\mathcal{O}} = \mathrm{Supp}(A - I) \cup \mathrm{Supp}(n - 1)$$

Let G be the unbounded component of $\overline{O}^c$

We call $D := \overline{G}^c$, and $\overline{D} \subset \Omega$

Now assume that the incident field v does not scatter, this is

$$w \equiv 0 \quad \text{in} \quad \mathbb{R}^d \setminus \overline{D}$$

Non-scattering wave number

Given the inhomogeneous media (A, n, D), we say k is a non-scattering wave number, if this problem has a solution

$$(*) \qquad \nabla \cdot A\nabla w + k^2 n w = \nabla \cdot (I - A)\nabla v_g + k^2(1-n)v_g \qquad \text{in } \mathbb{R}^d$$

$$w = 0 \text{ in } \mathbb{R}^d \setminus \overline{D} \text{ and}$$

$$v_g(x) = \int_{\mathbb{S}} e^{ikx\cdot\hat{y}} g(\hat{y})\, ds_{\hat{y}} \qquad (\Delta v_g + k^2 v_g = 0 \quad \text{in } \mathbb{R}^d)$$

Or $(*)$ satisfied in D together with conditions (overdetermined)

$$w = 0 \qquad \text{and} \qquad \nu \cdot A\nabla w = \nu \cdot (I - A)\nabla v_g \qquad \text{on } \partial D$$

Transmission Eigenvalues

Given the inhomogeneous media (A, n, D), we say k is a transmission eigenvalue, if there is nontrivial w and v satisfying

$$\Delta v + k^2 v = 0 \qquad \text{in } D$$

$$\nabla \cdot A \nabla w + k^2 n w = \nabla \cdot (I - A) \nabla v + k^2 (1 - n) v \qquad \text{in } D$$

$$w = 0 \qquad \text{and} \qquad \nu \cdot A \nabla w = \nu \cdot (I - A) \nabla v \qquad \text{on } \partial D$$

$u := w + v$

$$\Delta v + k^2 v = 0 \qquad \text{and} \qquad \nabla \cdot A \nabla u + k^2 n u = 0 \qquad \text{in } D$$

$$u = v \qquad \text{and} \qquad \nu \cdot A \nabla u = \nu \cdot \nabla v \qquad \text{on } \partial D$$

State of the Art of TEP - General Media

- Discreteness, completeness of eigenfunction, Weyl's asymptotic:
  $\partial D \in C^2$, $A, n \in C^1(\overline{D})$ and for $x \in \partial D$ and every unite $\xi \perp \nu$

  $$(A(x)\nu \cdot \nu)(A(x)\xi \cdot \xi) - (A(x)\nu \cdot \xi)^2 \neq 1 \quad \text{and} \quad (A(x)\nu \cdot \nu)n(x) \neq 1$$

  the first condition is equivalent to the Agmon, Douglis and Nirenberg complementing condition

  H.M. Nguyen-QH Nguyen (2021), H.M. Nguyen-J. Fornerod (2022)

- Existence of real TE: ∂D, Lipschitz, and A and n in $L^\infty(D)$. There exists an infinite sequence of real TE $\{k_j > 0\}$ accumulating at ∞, if $A - I$ and $n - 1$ are one sign (same or opposite) uniformly in D.
  Cakoni-Kirsch (2010)

- Location of TE: For ∂D in C^∞, $A = aI$, $a, n \in C^\infty(\overline{D})$ and the above contrast condition on ∂D, unfortunately the TEs do not have uniformly bounded imaginary part. Vodev (2015),(2018).

Case 2: $A \neq I$

k is a non-scattering wave number, if there exists a nontrivial v such that

$$\Delta v + k^2 v = 0 \qquad \text{in } \Omega$$

$$\nabla \cdot A \nabla w + k^2 n w = \nabla \cdot (I - A)\nabla v + k^2(1 - n)v \qquad \text{in } D$$

$$w = 0 \quad \text{and} \quad \nu \cdot A\nabla w = \nu \cdot (I - A)\nabla v \quad \text{on } \partial D$$

(Cakoni-Vogelius-Xiao, 2023)

We prove the same type of regularity result for an anisotropic inhomogeneity to be non-scattering, provided

$$\partial D \text{ is } C^{1,\mu} \qquad \text{and} \qquad \nu \cdot (A - I)\nabla v(z) \neq 0$$

F. Cakoni, M. Vogelius and J. Xiao (2023), Notes on the regularity of non-Scattering anisotropic Inhomogeneities, *Arch. Rat. Mech. Anal.*

Non-scattering Inhomogeneities with Corners

The case of curvilinear polygonal in $\mathbb{R}^2$ with $A = aI$ is analyzed by CGO solutions. There are inconclusive exceptional angles.

F. Cakoni and J. Xiao (2021) On corner scattering for operators of divergence form and applications to inverse scattering, *Anal. & PDEs*.

Example of a corner that does not scatter

Take $a = n \neq 1$ positive constants in D.

Observation: k is a transmission eigenvalue if and only if k^2 is either Dirichlet or Neumann eigenvalue for $-\Delta$ in D.

Now consider in particular $D := (0, 1) \times (0, 1)$. The n Dirichlet eigenpair

$$(p^2 + q^2)\pi^2, \quad \psi(x, y) := \sin(p\pi x)\sin(q\pi y), \; p, q \in \mathbb{N}$$

yield the corresponding transmission eigenfunction $(u, v) := (\psi, a\psi)$. Note ψ and $\nabla\psi$ vanishes at the corner.

Anisotropic Media

(A, n, D) be the push-forward of $(I, 1, D)$ under sufficiently smooth diffeomorphism $\Phi : D \to D$, with $\Phi = I$ on ∂D

$$A = \frac{D\Phi D\Phi^\top}{|\det D\Phi|} \circ \Phi^{-1} \quad \text{and} \quad n = \frac{1}{|\det D\Phi|} \circ \Phi^{-1} .$$

- Any k is a non-scattering wave number for any incident field. Transmission eigenvalues for this (A, n, D) are not discrete.

  A, n and D violate the sufficient conditions of discreteness of transmission eigenvalues.

- To understand why this construction does not contradict our non-scattering result, one must understand that if ∂D is not of class $C^{\ell+1,\mu}$ near z, then either Range$(A - I)(z) \perp \nu(z)$ or A, n fails to be $C^{\ell+1,\mu}$, $C^{\ell,\mu}$ near z.