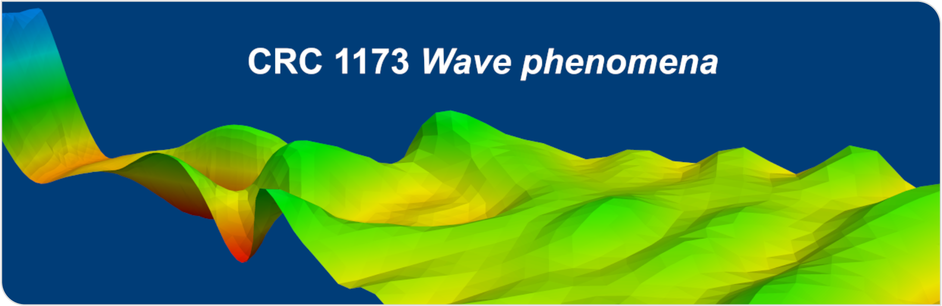
Karlsruhe Institute of Technology

phenomena

An Introduction to Inverse Problems

Andreas Kirsch, Athens | June 2023



Outline of the Course

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(A) Introduction and Examples
- Heat equation
- CT and Radon transform
- Impedance tomography
- Inverse scattering problem

(B) Ill-Posedness and Regularization
- Tikhonov Regularization
- Iterative regularization techniques
- Remarks

(C) Inverse Scattering theory
- Uniqueness
- Iterative methods
- Factorization method

I am following my monograph An Introduction to the Mathematical Theory of Inverse Problems, 3rd edition, 2020.

(A) Introduction and Examples

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We begin with important and classical examples:

Example A (Backwards heat equation) One-dimensional heat equation

$$\frac{\partial u(x,t)}{\partial t} = \frac{\partial^2 u(x,t)}{\partial x^2}, \quad (x,t) \in (0,\pi) \times \mathbb{R}_{>0},$$

with boundary and initial conditions

$$u(0,t) = u(\pi,t) = 0,\ t \geq 0, \quad u(x,0) = u_0(x),\ 0 \leq x \leq \pi.$$

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Separation of variables leads to the (formal) solution

$$u(x,t) = \sum_{n=1}^{\infty} a_n\, e^{-n^2 t} \sin(nx) \quad \text{with} \quad a_n = \frac{2}{\pi} \int_0^{\pi} u_0(y) \sin(ny) dy\,.$$

Direct problem: Given u_0 and $T > 0$, determine $u(\cdot, T)$.

Inverse problem: Measure $u(\cdot, T)$ and determine $u(\cdot, \tau)$ for given $\tau < T$.

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Recall:

$$u(x,T) \;=\; \frac{2}{\pi}\sum_{n=1}^{\infty}\int_0^{\pi} u(y,\tau)\sin(ny)dy\, e^{-n^2(T-\tau)}\sin(nx)\,.$$

Therefore, determine $v := u(\cdot, \tau)$ from integral equation

$$u(x,T) \;=\; \int_0^{\pi} k(x,y)\, v(y)\, dy, \quad 0 \le x \le \pi\,,$$

where

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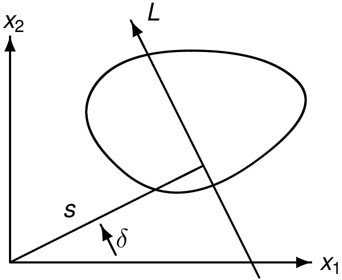
Inverse problem leads to solving a Fredholm integral equation of the first kind!

Example B (Computer tomography)

Consider a fixed plane through a human body. $\rho(x_1, x_2)$ is change of density at (x_1, x_2) and has to be determined from measurements of (attenuations of) intensities $I = I(L)$ of X-rays along lines L in the plane.



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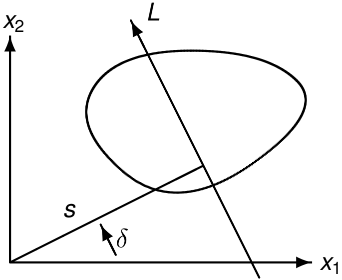

Parametrization of $L = L_{s,\delta}$:

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = s \begin{pmatrix} \cos\delta \\ \sin\delta \end{pmatrix} + t \begin{pmatrix} -\sin\delta \\ \cos\delta \end{pmatrix} \in \mathbb{R}^2, \quad t \in \mathbb{R}\,.$$

The attenuation of the intensity I is approximately described by $dI = -\gamma\,\rho\,I\,dt$ with some constant γ. Integration along the ray yields

$$\ln I_{s,\delta} \;=\; -\gamma \int\limits_{-\infty}^{\infty} \rho(s\cos\delta - t\sin\delta, s\sin\delta + t\cos\delta)\,dt\,.$$

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$$(R\rho)(s,\delta) := \int\limits_{-\infty}^{\infty} \rho(s\cos\delta - t\sin\delta, s\sin\delta + t\cos\delta)\, dt\,,\; (s,\delta) \in \mathbb{R} \times [0,\pi)\,.$$

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Special case: $\rho = \rho(r)$ radially symmetric, only vertical rays. This leads to Abel's integral equation for $z \mapsto \rho(\sqrt{R^2 - z})$:

$$V\big(\sqrt{R^2 - y}\big) \;=\; -\gamma \int_0^y \frac{\rho(\sqrt{R^2 - z})}{\sqrt{y - z}}\, dz, \quad 0 \le y \le R^2\,.$$

Example C (Impedance tomography)

Let $D \subset \mathbb{R}^2$ cross-section through body and $\gamma = \gamma(x_1, x_2)$ conductivity. Apply current distribution f on boundary ∂D. The potential u satisfies

$$\mathrm{div}(\gamma \nabla u) = 0 \text{ in } D\,, \quad \gamma\, \partial u / \partial \nu = f \text{ on } \partial D\,.$$

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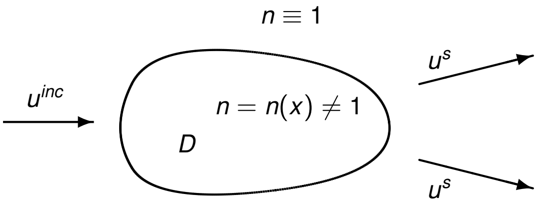
Inverse problem: Measure u on ∂D for many fluxes f and determine γ in D.

Example D (Inverse scattering problem)



Direct scattering problem: Given $n \in L^\infty(\mathbb{R}^3)$ such $D := \text{supp}(n-1)$ is bounded, the wave number $k > 0$, and the incident field $u^{inc}(x) = e^{ik\hat{\theta}\cdot x}$with $\hat{\theta} \in S^2$ (unit sphere), find the total field $u = u(x)$ with $\Delta u + k^2 nu = 0$ in $\mathbb{R}^3$ such that $u^s := u - u^{inc}$ satisfies a radiation condition for $|x| \to \infty$.

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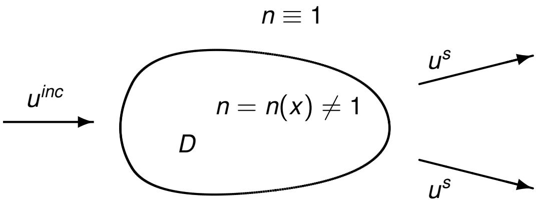

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Inverse scattering problem: Given u far away from D for all directions $\hat{\theta} \in S^2$, find n or at least the shape of $D = \text{supp}(n-1)$.

(B) Ill-Posedness and Regularization

Common properties of examples:
Direct problems are well-posed (in suitable function spaces and solution concepts); that is, existence, uniqueness, and continuous dependence on data holds.

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We model the problems by tripel (X, Y, K) where X and Y are normed spaces and $K : X \to Y$ a continuous linear or nonlinear operator (with domain $\mathcal{D}(K) \subset X$ and range $\mathcal{R}(K) \subset Y$).

Direct problem: Given $x \in X$, evaluate $K(x)$!
Inverse problem: Given $y \in \mathcal{R}(K)$ determine solution $x \in \mathcal{D}(K)$ with $K(x) = y$!
Ill-posedness (if K injective and surjective): K^{-1} is not continuous

Setting: X, Y Hilbert spaces, $K : X \to Y$ bounded linear operator. For simplicity: K is also one-to-one.

Theorem Let $\dim X = \infty$ and K compact. Then:
(a) $\mathcal{R}(K)$ is not closed in Y and $K^{-1} : \mathcal{R}(K) \to X$ is unbounded.

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(a) $\mathcal{R}(K)$ is not closed in Y and $K^{-1} : \mathcal{R}(K) \to X$ is unbounded.
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General Assumption: K linear, compact, and one-to-one, $\hat{y} \in \mathcal{R}(K)$ and $\hat{x} \in X$ solution of $K\hat{x} = \hat{y}$ and $y^\delta \in Y$ (not necessarily in $\mathcal{R}(K)$) with $\|y^\delta - \hat{y}\| \leq \delta$.

Aim: Solve (approximately) $Kx \approx y^\delta$ such that $x \approx \hat{x}$.

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Idea of regularization: Approximate $K^{-1} : \mathcal{R}(K) \to X$ by bounded operators $R_\alpha : Y \to X$ for (small) $\alpha > 0$ and set $x_{\alpha,\delta} := R_\alpha y^\delta$.

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(B1) Tikhonov regularization:

$$R_\alpha := (\alpha I + K^*K)^{-1} K^*$$

Lemma $R_\alpha K$ converges pointwise to the identity in X as $\alpha \to 0$; that is, $R_\alpha K x \to x$ as $\alpha \to 0$ for every $x \in X$.

Idea of proof: We have to show that $(\alpha I + K^*K)^{-1}K^*Kx \to x$ and calculate

$$(\alpha I + K^*K)^{-1}K^*Kx - x \;=\; -\,\alpha\,(\alpha I + K^*K)^{-1}x \;=\; -\alpha\, z_\alpha$$

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Multiplication with z_α:

$$\alpha\,\|z_\alpha\|^2 + \|Kz_\alpha\|^2 \;=\; (x, z_\alpha) \;\le\; \|x\|\,\|z_\alpha\|\,, \quad \text{thus} \quad \alpha\,\|z_\alpha\| \le \|x\|\,. \qquad (1)$$

$$(\alpha I + K^*K)^{-1}K^*Kx - x = -\alpha\, z_\alpha \;\Rightarrow\; \|(\alpha I + K^*K)^{-1}K^*K - I\| \le 1\,. \qquad (2)$$

Special case: $x = K^*u \in \mathcal{R}(K^*)$. With (1): $(x, z_\alpha) = (u, Kz_\alpha) \le \|u\|\,\|Kz_\alpha\|$, thus (1) has the form:

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thus $\alpha\|z_\alpha\|^2 \le \|u\|^2$ and thus $\alpha\|z_\alpha\| \le \sqrt{\alpha}\|u\|$. From (2) we get

$$\|(\alpha I + K^*K)^{-1}K^*Kx - x\| \;\le\; \sqrt{\alpha}\,\|u\| \;\longrightarrow\; 0\,,\; \alpha \to 0\,.$$

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General case $x \in X$: Note that closure$\big(\mathcal{R}(K^*)\big) = \mathcal{N}(K)^\perp = X$, and use Theorem of Banach-Steinhaus. □

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Back to (approximate) solution of $\boxed{Kx \approx y^\delta}$. Set $x_{\alpha,\delta} := R_\alpha y^\delta$. Then

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If $\alpha(\delta) \to 0$ and $\delta^2/\alpha(\delta) \to 0$ as $\delta \to 0$ then $x_{\alpha(\delta),\delta} \to \hat{x}$ as $\delta \to 0$.
Order of convergence: If $\hat{x} = K^*u$ then $\|x_{\alpha,\delta} - \hat{x}\| \le \frac{\delta}{\sqrt{\alpha}} + \sqrt{\alpha}\|u\|$. Choose $\alpha := \alpha(\delta) = c\,\delta$ which yields

$$\|x_{\alpha(\delta),\delta} - \hat{x}\| \le \left(c^{-1/2} + c^{1/2}\|u\|\right)\sqrt{\delta}\,.$$

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If $\alpha(\delta) \to 0$ and $\delta^2/\alpha(\delta) \to 0$ as $\delta \to 0$ then $x_{\alpha(\delta),\delta} \to \hat{x}$ as $\delta \to 0$.
Order of convergence: If $\hat{x} = K^*u$ then $\|x_{\alpha,\delta} - \hat{x}\| \leq \frac{\delta}{\sqrt{\alpha}} + \sqrt{\alpha}\|u\|$. Choose $\alpha := \alpha(\delta) = c\,\delta$ which yields

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Analogously, if $\hat{x} = K^*Ku$ then choose $\alpha(\delta) := c\,\delta^{2/3}$ which yields

$$\|x_{\alpha(\delta),\delta} - \hat{x}\| \leq \left(c^{-1/2} + 2c\|u\|\right)\delta^{2/3}\,.$$

So far: $R_\alpha K x \to x$ for all $x \in X$ and $\|R_\alpha Kx - x\| \leq c\sqrt{\alpha}$ for $x \in \mathcal{R}(K^*)$.
Furthermore, $\|R_\alpha\| \leq \frac{1}{\sqrt{\alpha}}$ as seen from previous arguments:
$z_\alpha := R_\alpha u = (\alpha I + K^*K)^{-1}K^*u$ is previous definition for $x = K^*u$, thus $\sqrt{\alpha}\|z_\alpha\| \leq \|u\|$.
Analogously: $\|R_\alpha Kx - x\| \leq c\,\alpha^{2/3}$ for $x \in \mathcal{R}(K^*K)$.

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Disadvantage for this a-priori choice: $\|u\|$ not known in advance!

Better is a-posteriori choice by discrepancy principle: Choose α such that

$$\|Kx_{\alpha,\delta} - y^\delta\| \; = \; \delta \quad \text{where} \quad x_{\alpha,\delta} = R_\alpha y^\delta \, . \tag{3}$$

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(d) In applications (K integral operator) the conditions $\hat{x} \in \mathcal{R}(K^*)$ or $\hat{x} \in \mathcal{R}\big((K^*K)^m\big)$ are smoothness assumptions on $\hat{x}$ combined with compatibility conditions.

(B2) Iterative regularization techniques

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Consider again the equation $Kx = y^\delta$. Assume that K^* is one-to-one; that is, K has dense range. Rewrite $Kx = y^\delta$ as equivalent fixpoint equation in the form $x = x - a\,K^*(Kx - y^\delta)$ with some parameter $a > 0$ and iterate:

$$x_{m+1,\delta} \;= x_{m,\delta} \;-\; a\,K^*\big(Kx_{m,\delta} - y^\delta\big), \quad m = 0, 1, 2, \ldots,$$

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with $x_{0,\delta} = 0$. Then $x_{m,\delta} = R_m y^\delta$ where $R_m : Y \to X$ is given by

$$R_m \;:=\; a \sum_{k=0}^{m-1} (I - a\,K^*K)^k K^* \quad \text{for } m = 1, 2, \ldots\ . \tag{4}$$

(Proof by induction with respect to m.) This is Landweber iteration and is the gradient method (with step size $a > 0$) corresponding to the minimization of $J(x) = \|Kx - y^\delta\|^2$.

Instead of discrepancy principle one uses the following stopping rule. Let $r > 1$ be fixed with $r\delta < \|y^\delta\|$. Let $m(\delta) \in \mathbb{N}$ such that

$$\|Kx_{m(\delta),\delta} - y^\delta\| \;\leq\; r\delta \;<\; \|Kx_{m,\delta} - y^\delta\| \quad \text{for all } m = 0, \ldots, m(\delta) - 1\,. \tag{5}$$

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Theorem Let $0 < a < 1/\|K\|^2$. Then $\lim_{m\to\infty} Kx_{m,\delta} = y^\delta$ which implies that there exists $m(\delta)$ with (5). If $\hat{x} = (K^*K)^{\sigma/2}z \in \mathcal{R}\big((K^*K)^{\sigma/2}\big)$ for some $z \in X$ and $\sigma > 0$ we have the estimate

$$\big\|x_{m(\delta),\delta} - \hat{x}\big\| \ \leq \ c\,\|z\|^{1/(\sigma+1)}\,\delta^{\sigma/(\sigma+1)} \tag{6}$$

for some $c > 0$ independent of δ.

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Conjugate gradient method: $x_0 = 0, \quad p_0 = -K^* y^\delta$.

$$x_{m+1} = x_m - t_m p_m\,, \qquad t_m = \frac{(Kx_m - y^\delta, Kp_m)}{\|Kp_m\|^2}\,,$$

$$p_{m+1} = K^*(Kx_{m+1} - y^\delta) + \gamma_m p_m\,, \qquad \gamma_m = \frac{\|K^*(Kx_{m+1} - y^\delta)\|^2}{\|K^*(Kx_m - y^\delta)\|^2}\,,$$

$m = 0, 1, \dots$. With the same stopping rule as above the same theorem holds.

(B3) Remarks

- There exist extensions of Tikhonov's method (iterated Tikhonov's method) and the discrepancy principle to extend the order optimality $\mathcal{O}(\delta^{2/3})$ and $\mathcal{O}(\sqrt{\delta})$, respectively.

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- Construction of regularization operators R_h by discretization; that is replace $K : X \to Y$ by $K_h : X_h \to Y_h$ with finite dimensional X_h, Y_h.

(C) Inverse Scattering Theory

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Recall the model for the scattering problem:

Total field u is sum of incident field u^{inc} and scattered field u^s; that is:
$u = u^{inc} + u^s$ satisfies the Helmholtz equation

$$\Delta u + k^2 n\, u = 0 \quad \text{in } \mathbb{R}^3\,,$$

and u^s satisfies Sommerfeld's radiation condition (SRC)

$$\frac{\partial u^s(x)}{\partial r} - iku^s(x) \;=\; \mathcal{O}\big(r^{-2}\big)\,, \quad r = |x| \to \infty\,,$$

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(a) Plane wave of direction $\hat{\theta} \in S^2$: $\qquad u^{inc}(x) \;=\; e^{ik\hat{\theta}\cdot x}\,, \quad x \in \mathbb{R}^3\,.$

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(b) Spherical wave with source point $z \in \mathbb{R}^3$ (fundamental solution)

$$\Phi(x,z) \;:=\; \frac{\exp(ik|x-z|)}{4\pi|x-z|}\,, \quad x \neq z\,.$$

The direct problem

$$\Delta u + k^2 n u = 0 \text{ in } \mathbb{R}^3 , \quad u^s := u - u^{inc} \text{ satisfies SRC.}$$

For $n \in L^\infty(\mathbb{R}^3)$ where $q := n - 1$ has bounded support the solution is searched for in (local) Sobolev space $H^2_{loc}(\mathbb{R}^3)$.

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Theorem: There exists at most one solution of the direct scattering problem (uniqueness).

Proof is based on Lemma of Rellich and unique continuation:

Lemma of Rellich: For $k > 0$ (real valued) and $\Delta u + k^2 u = 0$ for $|x| > R_0$ it holds that:

$$\lim_{R\to\infty} \int_{|x|=R} |u|^2 ds = 0 \quad \text{implies} \quad u = 0 \text{ for } |x| > R_0\,.$$

Unique Continuation: Let $u \in H^2_{loc}(\mathbb{R}^3)$ satisfy $\Delta u + k^2 n u = 0$ in $\mathbb{R}^3$. If $u = 0$ on some open set then u vanishes everywhere.

Proof of Uniqueness

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Uniqueness of direct problem: Assume u is difference of two solutions. Then $\Delta u + k^2 n u = 0$ in $\mathbb{R}^3$ and u satisfies the SRC. Then:

$$\int\limits_{|x|=R} \left| \frac{\partial u}{\partial r} - iku \right|^2 ds \;=\; \int\limits_{|x|=R} \left| \frac{\partial u}{\partial r} \right|^2 + k^2 |u|^2 ds \;+\; 2k \operatorname{Im} \int\limits_{|x|=R} u \, \frac{\partial \overline{u}}{\partial r} \, ds$$

The left hand side tends to zero by the SRC. Green's theorem yields

$$\int\limits_{|x|=R} u \, \frac{\partial \overline{u}}{\partial r} \, ds \;=\; \int\limits_{B_R} \left[|\nabla u|^2 + u \, \Delta \overline{u} \right] dx \;=\; \int\limits_{B_R} \left[|\nabla u|^2 - k^2 n \, |u|^2 \right] dx$$

and this is real valued. Therefore, $\int_{|x|=R} |u|^2 ds \to 0$ as $R \to \infty$.

Rellich's Lemma and unique continuation imply $u = 0$ in $\mathbb{R}^3$. □

Existence

Existence is based on volume potential for fundamental solution Φ.

Theorem: For $\varphi \in L^2(D)$ the potential

$$v(x) \; = \int_D \varphi(y)\, \Phi(x,y)\, dy\,, \quad x \in \mathbb{R}^3\,,$$

is the only radiating solution $v \in H^2_{loc}(\mathbb{R}^3)$ of $\Delta v + k^2 v = -\varphi$ in $\mathbb{R}^3$.

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Rewrite $\Delta u + k^2 n u = 0$ as $\Delta u + k^2 u = -k^2 q u$ where $q := n - 1$, thus also $\Delta u^s + k^2 u^s = -k^2 q u$, thus by theorem:

$$u(x) - u^{inc}(x) \;=\; u^s(x) \;=\; k^2 \int_D q(y)\,u(y)\,\Phi(x,y)\,dy\,, \quad x \in \mathbb{R}^3\,.$$

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Theorem: For every $n \in L^\infty(D)$ such that $q := n - 1$ is supported in D there exists a unique solution $u \in H^2_{loc}(\mathbb{R}^3)$ of the direct scattering problem. $u|_D$ solves the Lippmann-Schwinger integral equation.

Far Field Pattern

Recall Lippmann-Schwinger integral equation

$$u(x) - u^{inc}(x) \;=\; u^s(x) \;=\; k^2 \int_D q(y)\, u(y)\, \Phi(x,y)\, dy\,, \quad x \in \mathbb{R}^3\,.$$

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Asymptotic behavior $\Phi(x,z) = \frac{\exp(ik|x|)}{4\pi|x|} e^{-ik\hat{x}\cdot z} + \mathcal{O}(1/|x|^2)$ yields

$$u^s(x) \; = \; \frac{\exp(ik|x|)}{4\pi|x|}\, u^\infty(\hat{x}) \; + \; \mathcal{O}(1/|x|^2)\,, \quad |x| \to \infty\,,$$

uniformly wrt $\hat{x} := x/|x| \in S^2$ with far field pattern

$$u^\infty(\hat{x}) \; = \; k^2 \int_D q(y)\, u(y)\, e^{-ik\hat{x}\cdot y}\, dy\,, \quad \hat{x} \in S^2\,.$$

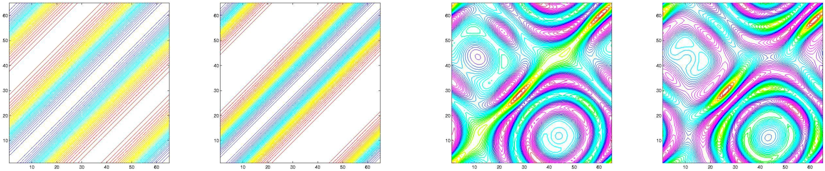
For $u^{inc}(x) = \exp(ik\hat{\theta}\cdot x)$ we have $u^\infty = u^\infty(\hat{x},\hat{\theta})$.

The Inverse Scattering Problem

Inverse scattering problem: Determine (properties of) the contrast $q(x) = n(x) - 1$ from the knowledge of $u^\infty(\hat{x}, \hat{\theta})$ for all $\hat{x}, \hat{\theta} \in S^2$!

2D-Example: Here $\hat{x}, \hat{\theta} \in S^1 \triangleq (0, 2\pi)$

Which contrast q corresponds to the following far fields $u^\infty(\phi, \theta)$, $\phi, \theta \in [0, 2\pi]$?



Re u^∞ Im u^∞ Re u^∞ Im u^∞

Uniqueness of Inverse Scattering Problem

Left example:

Theorem of Karp: If $u^\infty(\hat{x},\hat{\theta}) = \psi(\hat{x}\cdot\hat{\theta})$ for all $\hat{x},\hat{\theta} \in S^2$, then q is radially symmetric; that is, $q(x) = f(|x|)$ for some function $f \in L^\infty(\mathbb{R}_{>0})$. In particular, the support of q is a ball.

Uniqueness of Inverse Scattering Problem

Left example:

Theorem of Karp: If $u^\infty(\hat{x},\hat{\theta}) = \psi(\hat{x}\cdot\hat{\theta})$ for all $\hat{x},\hat{\theta} \in S^2$, then q is radially symmetric; that is, $q(x) = f(|x|)$ for some function $f \in L^\infty(\mathbb{R}_{>0})$. In particular, the support of q is a ball.

Uniqueness of the inverse scattering problem:

Theorem The far field patterns $u^\infty(\hat{x},\hat{\theta})$ determine n uniquely; that is, if $n_j \leftrightarrow u_j^\infty(\hat{x},\hat{\theta})$ for $j = 1, 2$, then:

$$u_1^\infty(\hat{x},\hat{\theta}) = u_2^\infty(\hat{x},\hat{\theta}) \text{ for all } \hat{x},\hat{\theta} \in S^2 \implies n_1 = n_2\,.$$

In $\mathbb{R}^3$: Nachman (1988), Novikov (1988), Ramm (1988)

In $\mathbb{R}^2$: Bukhgeim (2008)

Drossos Gintides will talk on this topic!

Reconstruction Techniques

(a) Linearization, e.g. Born approximation: Recall L-S-eqn:

$$u(x) \;=\; u^{inc}(x) \;+\; k^2 \int_D q(y)\, u(y)\, \Phi(x,y)\, dy\,, \quad x \in D\,.$$

Iteration converges if norm of operator is less than 1. First iteration:

$$\begin{aligned}
u_B(x) &= u^{inc}(x) \;+\; k^2 \int_D q(y)\, u^{inc}(y)\, \Phi(x,y)\, dy\,, \quad x \in \mathbb{R}^3\,,\\
u_B^s(x,\hat\theta) &= k^2 \int_D q(y)\, u^{inc}(y,\hat\theta)\, \Phi(x,y)\, dy\,, \quad x \in \mathbb{R}^3\,,\\
u_B^\infty(\hat x,\hat\theta) &= k^2 \int_D q(y)\, u^{inc}(y,\hat\theta)\, e^{-ik\hat x\cdot y} dy\\
&= k^2 \int_D q(y)\, e^{ik(\hat\theta-\hat x)\cdot y} dy = k^2\, \hat q\big(k(\hat x-\hat\theta)\big)\,,\ \hat x,\hat\theta \in S^2.
\end{aligned}$$

Determine q from Fourier transform on ball; that is, for $k(\hat x-\hat\theta) \in \{z \in \mathbb{R}^3 : |z| \le 2k\}$. Problem is linear and ill-posed!

(b) Iterative methods to determine contrast function q: Define mapping $\mathcal{T} : L^\infty(D) \to L^2(S^2 \times S^2)$, $q \mapsto u^\infty$. Apply iterative method to solve $\boxed{\mathcal{T}(q) = f}$ for q where $f = f(\hat{x}, \hat{\theta})$ is given (measured) far field pattern.

Possible methods: Newton-type methods, gradient-type methods, second order methods.

Derivative: $\mathcal{T}'(q)h = v^\infty$ where v is radiating solution of $\Delta v + k^2(1+q)v = -k^2 hu$. Derivative $\mathcal{T}'(q)$ is compact and one-to-one!

Advantages: Very general, accurate, incorporation of a priori information possible.

Disadvantages: “Expensive”, only local convergence is expected, no rigorous convergence result known.

(c) Sampling Methods. They determine **only support** D **of** q. Choose set of *sampling objects*, e.g. points $z \in \mathbb{R}^3$, and construct binary criterium which uses only the data u^∞ to decide whether or not z belongs to D.

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Disadvantages: Needs $u^\infty(\hat{x}, \hat{\theta})$ for many (in theory: all) $\hat{x}, \hat{\theta}$, no incorporation of a-priory information possible, very sensitive to noise.

Factorization Method

Factorization Method determines only support of $q := n - 1$! Values of $q \in L^\infty(\mathbb{R}^3)$ do not have to be known in advance.

Define far field operator $F : L^2(S^2) \to L^2(S^2)$ by

$$(Fg)(\hat{x}) \;=\; \int_{S^2} u^\infty(\hat{x},\hat{\theta})\, g(\hat{\theta})\, ds(\hat{\theta})\,, \quad \hat{x} \in S^2\,.$$

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- F is compact.
- If q is real-valued then F is normal; that is, $F^* F = F F^*$, and even: $\mathcal{S} := I + \frac{ik}{2\pi} F$ is unitary (=scattering matrix).
- F is one-to-one if k^2 is not an interior transmission eigenvalue;

  that is, $\Delta u + k^2(1+q)\, u = 0$ in D, $\quad \Delta w + k^2 w = 0$ in D,

  $$u = w \text{ on } \partial D, \quad \partial u/\partial\nu = \partial w/\partial\nu \text{ on } \partial D,$$

  implies $u = w = 0$ in D. (Fioralba Cakoni will talk on this topic!)

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The scattered field satisfies

$$\Delta u^s + k^2(1+q)u^s = -k^2 q\, u^{inc} \quad \text{in } \mathbb{R}^3 \,.$$

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Theorem: $F : L^2(S^2) \to L^2(S^2)$ has factorization $\boxed{F = H^* T H}$ where $H : L^2(S^2) \to L^2(D)$ is defined as

$$(Hg)(x) \;=\; \int\limits_{S^2} u^{inc}(x,\hat{\theta})\, g(\hat{\theta})\, ds(\hat{\theta}) \;=\; \int\limits_{S^2} e^{ikx\cdot\hat{\theta}}\, g(\hat{\theta})\, ds(\hat{\theta}) \,,\; x \in D \,,$$

and $T : L^2(D) \to L^2(D)$ is defined as $Tf = k^2 q(f+v)$ where v is the radiating solution of

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Then $z \in D$ if, and only if, $\phi_z \in \mathcal{R}(H^*)$.

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Proof:
$$(H^*\varphi)(\hat{x}) = \int_D \varphi(y)\, e^{-ik\hat{x}\cdot y}\, dy \;\stackrel{?}{=}\; e^{-ik\hat{x}\cdot z}\,,\; \hat{x} \in S^2\,.$$

This is equivalent to (because complement of D is connected)

$$(*) \qquad \int_D \varphi(y)\,\Phi(x,y)\,dy \;=\; \Phi(x,z)\,, \quad x \notin (D \cup \{z\})\,.$$

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$z \in D$: Choose $\tilde{\Phi} \in C^\infty(\mathbb{R}^3)$ with $\tilde{\Phi}(x) = \Phi(x,z)$ outside of D and define φ by $\int_D \varphi(y)\,\Phi(\cdot,y)\,dy \;=\; \tilde{\Phi}$ in $\mathbb{R}^3$; that is, $-\varphi = \Delta\tilde{\Phi} + k^2\tilde{\Phi}$.

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$z \notin D$: $(*)$ can not have a solution! (Left hand side bounded, right hand side unbounded for $x \to z$.)

Recall: $\boxed{F = H^* \, T \, H}$ and $\boxed{z \in D \Longleftrightarrow \phi_z \in \mathcal{R}(H^*)}$.

Goal: Express range $\mathcal{R}(H^*)$ by known operator F!

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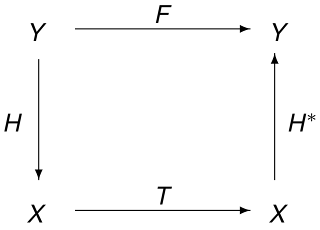
General situation for Hilbert spaces X, Y:



Theorem: If $T : X \to X$ is selfadjoint and coercive; that is,

$$\langle \psi, T\varphi \rangle = \langle T\psi, \varphi \rangle \,, \qquad \langle \varphi, T\varphi \rangle \geq c \|\varphi\|_X^2 \quad \text{for all } \psi, \varphi \in X \,,$$

then
$$\mathcal{R}(H^*) \;=\; \mathcal{R}(F^{1/2}) \,.$$

Theorem: Let $F = H_1^* T_1 H_1 = H_2^* T_2 H_2$ such that $T_j : X_j \to X_j$ is coercive in the sense that

$$\left|\langle T_j\varphi, \varphi\rangle\right| \;\geq\; c\|\varphi\|_{X_j}^2 \quad \text{for all } \varphi \in \mathcal{R}(H_j)\,,\; j = 1, 2\,.$$

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The proof follows from inf–condition (Kirsch) or a theorem of Nachman, Pävärinta, Teirilä: Let $H = H_1$ or H_2. Then

$$\begin{aligned}
\phi \in \mathcal{R}(H^*) \quad &\Longleftrightarrow \quad \exists c > 0 :\ |\langle\phi, \psi\rangle_Y|^2 \leq c|\langle F\psi, \psi\rangle_Y|\ \forall\, \psi \in Y \\
&\Longleftrightarrow \quad \inf\big\{|\langle F\psi, \psi\rangle_Y| : \langle\phi, \psi\rangle_Y = 1\big\} \;>\; 0 \\
&\Longleftrightarrow \quad \phi \perp \{\psi : \langle F\psi, \psi\rangle_Y = 0\}\ \big(= \mathcal{N}(H)\big) \quad \text{and} \\
&\qquad\quad\ \sup\big\{|\langle\phi, \psi\rangle_Y| : |\langle F\psi, \psi\rangle_Y| = 1\big\} \;<\; \infty
\end{aligned}$$

Theorem: Let $F = H^* T H : Y \to Y$ be one-to-one and such that $I + irF$ is unitary for some $r > 0$. Furthermore, let $T : X \to X$ be comp. perturb. of s.a. and coercive operator and $\operatorname{Im}\langle\varphi, T\varphi\rangle \neq 0$ for all $\varphi \in$ closure $\mathcal{R}(H)$ with $\varphi \neq 0$.

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$$|F|^{1/2}\psi = \sum_j \sqrt{|\lambda_j|}\,\langle\psi, \psi_j\rangle_Y\,\psi_j\,,$$

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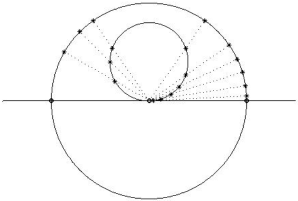

$$|\langle S\psi, \psi\rangle| = \left|\sum_j \frac{\lambda_j}{|\lambda_j|}\,\big|\langle\psi, \psi_j\rangle_Y\big|^2\right|$$

$$\geq c\|\psi\|_Y^2$$

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Combination of previous theorems:

Theorem: Let again $\phi_z(\hat{x}) = \exp(-ik\hat{x} \cdot z)$, $\hat{x} \in S^2$.

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Let $\{\lambda_j : j \in \mathbb{N}\} \subset \mathbb{C}$ be eigenvalues of (normal!) operator F with normalized eigenfunctions $\psi_j \in L^2(S^2)$ for $j \in \mathbb{N}$. Then:

$$z \in D \iff \sum_{j \in \mathbb{N}} \frac{|\langle \phi_z, \psi_j \rangle_{L^2}|^2}{|\lambda_j|} < \infty \iff \left[\sum_{j \in \mathbb{N}} \frac{|\langle \phi_z, \psi_j \rangle_{L^2}|^2}{|\lambda_j|} \right]^{-1} > 0\,.$$

Media with Absorption

Now $q \in L^\infty(\mathbb{R}^3)$ complex valued, $\operatorname{Im} q \geq 0$. Still $\boxed{F = H^* T H}$ but not normal anymore. Define

$$\begin{aligned}
\operatorname{Re} F &= \frac{1}{2}(F + F^*) = H^*(\operatorname{Re} T) H \\
\operatorname{Im} F &= \frac{1}{2i}(F - F^*) = H^*(\operatorname{Im} T) H \\
F_\# &=_{\text{def}} |\operatorname{Re} F| + \operatorname{Im} F
\end{aligned}$$

Then $F_\# = H^* \tilde{T} H$ with coercive $\tilde{T}$.

Theorem: $z \in D \iff \phi_z \in \mathcal{R}(F_\#^{1/2})$

Let $\{\lambda_j : j \in \mathbb{N}\} \subset \mathbb{R}$ be eigenvalues of (selfadjoint!) operator $F_\#$ with normalized eigenfunctions $\psi_j \in L^2(S^2)$ for $j \in \mathbb{N}$. Then:

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Extensions

Factorization method needs only far field patterns $u^\infty(\hat{x}, \hat{\theta})$ for all $\hat{x}, \hat{\theta} \in S^2$. If these are available, the method can be implemented.

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- Reduced data: $A \subset S^2$ (relative) open,data: $u^\infty(\hat{x}, \hat{\theta})$ for $\hat{x}, \hat{\theta} \in A$ (i.e. forward scattering): FM justified by using projection onto $L^2(A)$.

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- Point source incidence: $u^{inc}(x) = \Phi(x, z)$ for $z \in \Gamma$.
  Data: scattered fields $u^s(x, z)$ for $x, z \in \Gamma$. Factorization: $F = \overline{H^*}TH$, no range identity known, thus FM not justified!

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However, the method has to be justified for all models of wave propagation.

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- Obstacles D with boundary conditions:
  - Scattering by an arc (in $\mathbb{R}^2$) or screen (in $\mathbb{R}^3$): FM justified.
  - Scattering by impenetrable obstacle with Dirchlet-, Neumann-, impedance-, conductive boundary conditions: FM justified.
    Mixed boundary conditions ($D = D_1 \cup D_2$, Dirichlet bc on ∂D_1, Neumann bc on ∂D_2) not justified!

- Other models of wave propagation:
  - Anisotropic media, e.g. $\nabla \cdot (A \nabla u) + k^2 u = 0$
  - Electromagnetic wave propagation, modelled by Maxwell's equations
  - Elastic wave propagation, modelled by Navier's equations
  - Stokes problem
  - Hybrid model: elastic core in fluid
- Nonlinear Helmholtz equation
- Impedance tomography
- Periodic structures
- Wave guides

Numerical Simulations in $\mathbb{R}^2$

Karlsruhe Institute of Technology

Recall:

$$z \in D \iff \sum_{j\in\mathbb{N}} \frac{|\langle \phi_z, \psi_j \rangle_{L^2}|^2}{|\lambda_j|} < \infty$$

$$\iff w(z) = \left[\sum_{j\in\mathbb{N}} \frac{|\langle \phi_z, \psi_j \rangle_{L^2}|^2}{|\lambda_j|} \right]^{-1} > 0 \,.$$

Therefore, $\operatorname{sign}(w)$ is the characteristic function of D!

The following examples show plots of

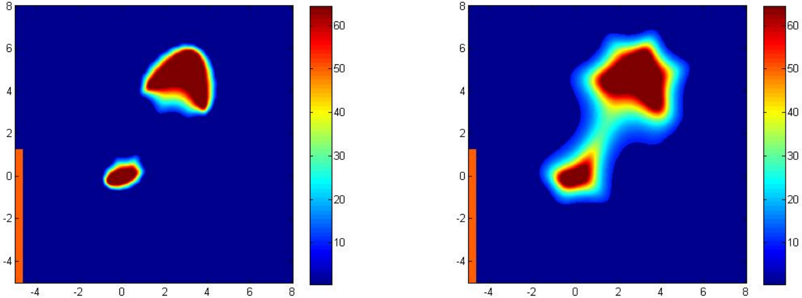
$$w_N(z) = \left[\sum_{j=1}^{N} \frac{|\langle \phi_z, \psi_j \rangle|^2}{|\lambda_j|} \right]^{-1}, \quad z \in \mathbb{R}^2 :$$

for $N = 32$ or $N = 36$, respectively.

Numerical Simulations

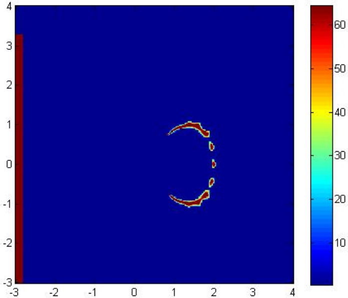
Karlsruhe Institute of Technology

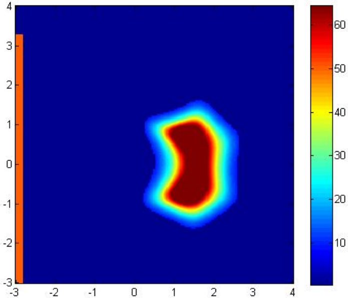
Dirichlet boundary conditions:



Numerical Simulations

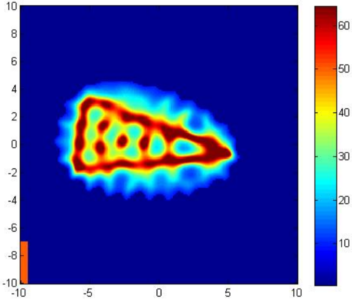
Scattering by an open arc:

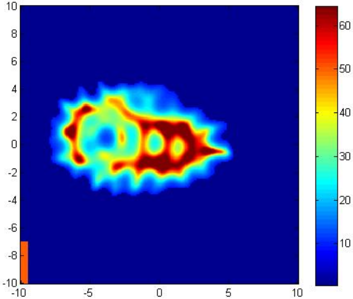




Numerical Simulations

Real data:



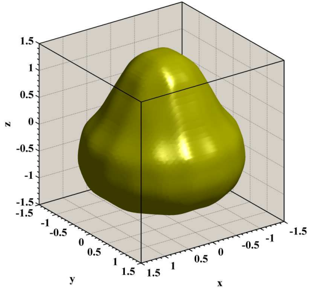


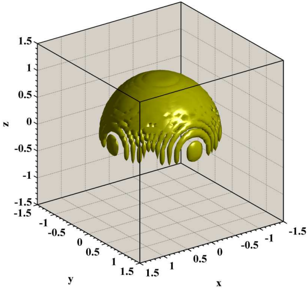
Numerical Simulations

3D-Example (joint work with A. Kleefeld): Scattering under conductive transmission conitions

$$\Delta u + k^2 u = 0 \quad \text{in } \mathbb{R}^3 \setminus \partial D\,,$$

$$u_+ = u_-\,, \quad \frac{\partial u_+}{\partial \nu} - \frac{\partial u_-}{\partial \nu} = \lambda\, u \quad \text{on } \partial D\,.$$





Thank you for your attention!