A dimension-reduction method for the numerical solution of various Cauchy problems in R^2

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Outline

- Introduction
- The two-step method
- Direct and inverse problems:
  - Elastodynamic problem
  - Wave equation
  - Heat equation
- Numerical implementation / results

Introduction

Example: Parabolic PDE

$$\begin{aligned} \frac{\partial u}{\partial t}(x,t) - \Delta u(x,t) &= 0, && x \in D \subset \mathbb{R}^2,\ t > 0, \\ u(x,0) &= 0, && x \in D, \\ u(x,t) &= f(x,t), && x \in \Gamma,\ t > 0, \end{aligned}$$

Introduction

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Methods

single-step Direct application of numerical scheme (FDM, integral equation methods).

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Methods

single-step: Direct application of numerical scheme (FDM, integral equation methods).

two-step: Split and threat differently time and space variables.

Introduction

Single-step method

We define the potentials

$$(\mathcal{S}_t\phi)(x,t) = \int_0^t \int_\Gamma \Phi(x-y, t-\tau)\phi(y,\tau)ds(y)d\tau,$$
$$(\mathcal{D}_t\phi)(x,t) = \int_0^t \int_\Gamma \frac{\partial}{\partial\nu(y)}\Phi(x-y, t-\tau)\phi(y,\tau)ds(y)d\tau,$$

for the fundamental solution Φ of the heat equation.

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The solution is represented by:

$$u(x,t) = (\mathcal{S}_t\partial_\nu u)(x,t) - (\mathcal{D}_t u)(x,t) \quad \text{(direct method)},$$

or by

$$u(x,t) = (\mathcal{S}_t\phi)(x,t) \quad \text{(indirect method)}.$$

Introduction

Two-step methods

- Finite Difference Method (FDM) together with Boundary Integral Equations (BIE) method - Rothe method
- Integral Transform (LT) together with BIE method

The two-step method

Rothe method

Step 1: Apply FDM w.r.t. time:

$$\frac{u_n(x) - u_{n-1}(x)}{h} = \Delta u_n(x), \quad in\ D,$$

where $u_n(x) = u(t_n, x)$, for the grid points $t_n = (n+1)h$, where $h = T/N$, for $n = 0, 1, ...N-1$.

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Step 2: Use BIE to derive

$$u_n(x) = \frac{1}{\pi}\int_\Gamma \frac{\partial}{\partial\nu(y)}\Phi(x-y)\phi_n(y)ds(y) + \frac{1}{2\pi h}\int_D \Phi(x-y)u_{n-1}(y)dy$$

The two-step method

Proposed method - Preliminaries

Normalized Laguerre polynomials

$$L_n(t) = \frac{1}{n!} e^t \frac{d^n}{dt^n}(t^n e^{-t}), \quad n = 0, 1, \dots$$

e.g. $L_0(t) = 1, \quad L_1(t) = 1 - t.$

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Properties

- $L_n' = -\sum_{m=0}^{n-1} L_m, \quad n = 1, 2, \ldots$
- $L_n(0) = 1, \quad L_n'(0) = -n, \quad n = 0, 1, \ldots$
- $\int_0^\infty e^{-t} L_n(t) L_m(t) dt = 0, \quad n \neq m$

The two-step method

Preliminaries

Let $u \in C^2$ bounded, we consider the expansion

$$u(t) = \sum_{n=0}^{\infty} u_n L_n(t),$$

where

$$u_n = \int_0^{\infty} e^{-t} u(t) L_n(t) dt, \quad n = 0, 1, \ldots$$

The two-step method

Preliminaries

Let $u \in C^2$ bounded, we consider the expansion

$$u(x,t) = \kappa \sum_{n=0}^{\infty} u_n(x) L_n(\kappa t),$$

where

$$u_n(x) = \int_0^{\infty} e^{-\kappa t} u(x,t) L_n(\kappa t) dt, \quad n = 0, 1, \ldots$$

The two-step method

Proposed method

Step 1: Apply Laguerre transform w.r.t. time:

$$\int_0^\infty e^{-\kappa t} L_n(\kappa t) \left(\Delta u(x,t) - \frac{\partial u}{\partial t}(x,t) \right) dt = \Delta u_n(x) - \kappa \sum_{m=0}^{n} u_m(x) + \cancel{u(x,0)},$$

for $n = 0, 1, \ldots$ Rewrite it as:

$$\Delta u_n(x) - \kappa u_n(x) = \kappa \sum_{m=0}^{n-1} u_m(x)$$

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The two-step method

- Initial boundary value problem:

$$\frac{\partial^\alpha}{\partial t^\alpha} u(x,t) = \mathcal{D}_x u(x,t), \quad \mathbb{R}^2 \supset D \times (0,\infty),$$

for $\alpha = 1, 2$, together with IC and BC.

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- Time-discretization:

$$u(x,t) = \kappa \sum_{n=0}^{\infty} u_n(x) L_n(\kappa t)$$

  resulting to a sequence of stationary problems.

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- Boundary integral equation method:

$$u_n(x) = \sum_{m=0}^{n} (\mathcal{S}_{n-m}\, \phi_m)(x),$$

for the unknown densities ϕ_m, defined on Γ.

The two-step method

Advantages

- Dimension reduction
- Avoid domain discretization and volume integrals
- Applicable in various cases
- Exponential convergence for exact data

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Drawbacks

- The fundamental sequence needs special treatment
- Working with singular integrals
- Shares the disadvantages of the BIE method (smooth boundary, good initial guess)

Hyperbolic case 1: the direct problem

Let $D \subset \mathbb{R}^2$ be bounded with C^2-smooth boundary Γ.

We consider the initial boundary value problem:

$$\begin{aligned} \frac{\partial^2 u}{\partial t^2}(x,t) - \Delta^* u(x,t) &= 0, && x \in \mathbb{R}^2 \setminus D,\ t > 0, \\ u(x,0) = \frac{\partial u}{\partial t}(x,0) &= 0, && x \in \mathbb{R}^2 \setminus D, \\ u(x,t) &= f(x,t), && x \in \Gamma,\ t > 0, \end{aligned} \tag{1}$$

where

$$\Delta^* u := c_s^2 \Delta u + (c_p^2 - c_s^2) \nabla\nabla \cdot u,$$

for the velocities $c_s = \sqrt{\mu/\rho}$, $c_p = \sqrt{(\lambda + 2\mu)/\rho}$, where ρ is the density, and λ and μ are the Lamé constants.

Hyperbolic case 1: the direct problem

The boundary function f satisfies

$$f(x,0) = \frac{\partial f}{\partial t}\,(x,0) = 0, \quad x \in \Gamma.$$

We impose the RC

$$u(x,t) \to 0, \quad \text{as} \quad |x| \to \infty,$$

uniformly to all directions $\frac{x}{|x|}$, and all $t \in [0,\infty)$.

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uniformly to all directions $\frac{x}{|x|}$, and all $t \in [0,\infty)$.

This problem is well-posed [V. Kupradze, 1979].

Hyperbolic case 1: time-discretization

We consider the expansion

$$u(x,t) = \kappa \sum_{n=0}^{\infty} u_n(x) L_n(\kappa t),$$

with Fourier-Laguerre coefficients

$$u_n(x) = \int_0^{\infty} e^{-\kappa t} L_n(\kappa t) u(x,t) dt, \quad n \in \mathbb{N}_0.$$

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$$u_n(x) = \int_0^{\infty} e^{-\kappa t} L_n(\kappa t) u(x,t)\, dt, \quad n \in \mathbb{N}_0.$$

Applying the Laguerre transform, we obtain

$$\begin{aligned} \Delta^* u_n(x) - \kappa^2 u_n(x) &= \sum_{m=0}^{n-1} \beta_{n-m} u_m(x), && x \in \mathbb{R}^2 \setminus D, \\ u_n(x) &= f_n(x), && x \in \Gamma, \\ u_n(x) &\to 0, && |x| \to \infty, \end{aligned} \tag{2}$$

for $n \in \mathbb{N}_0$, and $\beta_n = \kappa^2(n+1)$.

Hyperbolic case 1: time-discretization

Theorem

The sequence of stationary problems (2) *has at most one solution.*

Hyperbolic case 1: time-discretization

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Theorem

A sufficiently smooth function of the form

$$u(x,t) = \kappa \sum_{n=0}^{\infty} u_n(x) L_n(\kappa t),$$

is the solution of the initial BVP iff its Fourier-Laguerre coefficients $u_n,\ n \in \mathbb{N}_0$, *solve the system* (2).

Hyperbolic case 1: BIE

We introduce the sequence of functions

$$\Phi_n(\gamma, r) = K_0(\gamma r)\, v_n(\gamma, r) + K_1(\gamma r)\, w_n(\gamma, r), \quad n = 0, \ldots, N-1,$$

where K_0 and K_1 are the modified Hankel functions of order zero and one, respectively. The polynomials are given by

$$v_n(\gamma, r) = \sum_{m=0}^{\left[\frac{n}{2}\right]} a_{n,2m}(\gamma) r^{2m}, \quad w_n(\gamma, r) = \sum_{m=0}^{\left[\frac{n-1}{2}\right]} a_{n,2m+1}(\gamma) r^{2m+1},$$

where the coefficients $a_{n,m}$ satisfy the recurrence relations

$$a_{n,0}(\gamma) = 1, \quad n = 0, 1, \ldots, N-1,$$
$$a_{n,n}(\gamma) = -\frac{\gamma}{n}\, a_{n-1,n-1}(\gamma), \quad n = 1, 2, \ldots, N-1,$$

and

$$a_{n,m}(\gamma) = \frac{1}{2\gamma m}\left\{4\left[\frac{m+1}{2}\right]^2 a_{n,m+1}(\gamma) - \gamma^2 \sum_{k=m-1}^{n-1} (n-k+1) a_{k,m-1}(\gamma)\right\},$$

for $m = n-1, \ldots, 1$.

Hyperbolic case 1: BIE

Let

$$J(x) = \frac{xx^\top}{|x|^2}, \quad x \in \mathbb{R}^2 \setminus \{0\}.$$

Then, the sequence of matrices

$$E_n(x,y) = \Phi_{1,n}(|x-y|)I + \Phi_{2,n}(|x-y|)J(x-y)$$

are fundamental solutions, with

$$\Phi_{\ell,n}(r) = \frac{(-\ell)^{\ell-1}}{\kappa^2 r^2} \sum_{k=-2}^{2} \chi_{k,n} \left(\Phi_{n+k}(\tfrac{\kappa}{c_s}, r) - \Phi_{n+k}(\tfrac{\kappa}{c_p}, r) \right) + \frac{(-1)^{\ell-1}}{c_p^2} \Phi_n(\tfrac{\kappa}{c_p}, r) + \frac{\ell-1}{c_s^2} \Phi_n(\tfrac{\kappa}{c_s}, r), \quad \ell = 1,2.$$

Here, $\chi_{-2,n} = n(n-1)$, $\chi_{-1,n} = -4n^2$, $\chi_{0,n} = 2(3n^2+3n+1)$, $\chi_{1,n} = -4(n+1)^2$ and $\chi_{2,n} = (n+1)(n+2)$.

Hyperbolic case 1: BIE

We consider a sequence of single-layer potentials

$$U_n(x) = \frac{1}{2\pi} \sum_{m=0}^{n} \int_{\Gamma} E_{n-m}(x,y)\phi_m(y)\, ds(y), \quad x \in \mathbb{R}^2 \setminus D,$$

for the unknown densities $\phi_m \in C(\Gamma)$.

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for the unknown densities $\phi_m \in C(\Gamma)$.

Theorem

The sequence of single-layer potentials is a solution of (2) *provided that their densities satisfy*

$$\frac{1}{2\pi} \int_\Gamma E_0(x,y)\phi_n(y)\, ds(y) = f_n(x) - \frac{1}{2\pi} \sum_{m=0}^{n-1} \int_\Gamma E_{n-m}(x,y)\phi_m(y)\, ds(y), \quad x \in \Gamma.$$

Hyperbolic case 1: BIE

Theorem

For any sequence $f_n \in C^{1,\alpha}(\Gamma)$, the system of BIE admits a unique solution $\phi_n \in C^{0,\alpha}(\Gamma)$.

Proof.

The proof is by induction. For $n = 0$, the integral equation

$$\frac{1}{2\pi} \int_\Gamma E_0(x, y)\phi_0(y)\, ds(y) = f_0(x), \quad x \in \Gamma,$$

has a unique solution $\phi_0 \in C^{0,\alpha}(\Gamma)$ [R. Kress, 2014]. □

Hyperbolic case 1: quadrature rules

We consider the parametrization

$$\Gamma = \{x(s) = (x_1(s), x_2(s)) : 0 \leq s \leq 2\pi\},$$

where $x : \mathbb{R} \to \mathbb{R}^2$ is C^1 and 2π–periodic with $|x'(s)| > 0$, $\forall s$.

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Then, we obtain

$$\frac{1}{2\pi}\int_0^{2\pi} H_0(s,\tau)\psi_n(\tau)\,d\tau$$
$$= f_n(x(s)) - \frac{1}{2\pi}\sum_{m=0}^{n-1}\int_0^{2\pi} H_{n-m}(s,\tau)\psi_m(\tau)\,d\tau, \quad 0 \le s \le 2\pi,$$

Here, $\psi_n(s) := |x'(s)|\,\phi_n(x(s))$, and $H_n(s,\tau) := E_n(x(s), x(\tau))$.

Hyperbolic case 1: quadrature rules

The kernels H_n are decomposed as

$$H_n(s,\tau) = H_n^1(s,\tau)\ln\left(\frac{4}{e}\sin^2\frac{s-\tau}{2}\right) + H_n^2(s,\tau),$$

for some analytic functions H_n^1 and H_n^2.

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for some analytic functions H_n^1 and H_n^2.

We approximate [R. Kress, 2014]

$$\frac{1}{2\pi}\int_0^{2\pi} f(\tau)\ln\left(\frac{4}{e}\sin^2\frac{s_j-\tau}{2}\right)d\tau \approx \sum_{k=0}^{2M-1} R_{|j-k|}\, f(s_k),$$

$$\frac{1}{2\pi}\int_0^{2\pi} f(\tau)\,d\tau \approx \frac{1}{2M}\sum_{k=0}^{2M-1} f(s_k)$$

with the weights

$$R_j := -\frac{1}{2M}\left\{1 - 2\sum_{m=1}^{M-1}\frac{1}{m}\cos\frac{mj\pi}{M} + \frac{(-1)^j}{M}\right\}, \quad j = 0, \dots, 2M-1.$$

Hyperbolic case 1: collocation method

We solve

$$\sum_{k=0}^{2M-1} \left\{ R_{|j-k|} H_0^1(s_j, s_k) + \frac{1}{2M} \, H_0^2(s_j, s_k) \right\} \psi_{n,M}(s_k) = G_{n,M}(s_j),$$

for $j = 0, \ldots, 2M-1$, where

$$G_{n,M}(s_j) = g_n(s_j) - \sum_{m=0}^{n-1} \sum_{k=0}^{2M-1} \left\{ R_{|j-k|} H_{n-m}^1(s_j, s_k) + \frac{1}{2M} \, H_{n-m}^2(s_j, s_k) \right\} \psi_{m,M}(s_k).$$

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For analytic boundary values, we get

$$\|\psi_n - \psi_{n,M}\|_\infty \leq C_n e^{-\sigma M}$$

for positive constants C_n and σ.

Hyperbolic case 1: collocation method

Given the approximate solution $\psi_{n,M}$, we compute the coefficients

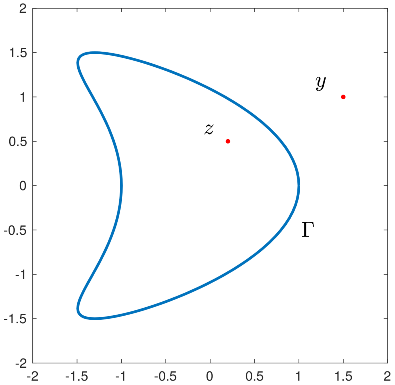
$$\tilde{u}_{n,M}(x) = \frac{1}{2M} \sum_{m=0}^{n} \sum_{k=0}^{2M-1} E_{n-m}(x, x(s_k)) \psi_{m,N}(s_k).$$

and then we obtain

$$u_{N,M}(x,t) = \kappa \sum_{n=0}^{N-1} \tilde{u}_{n,M}(x) L_n(\kappa t), \quad x \in \mathbb{R}^2 \setminus D.$$

Hyperbolic case 1: numerical results

Example 1: We set $\lambda = 2$, $\mu = 1$ and $\rho = 1$.



We choose a source point $z \in D$, and we define

$$f_n(x) = [E_n(x, z)]_1, \quad x \in \Gamma.$$

Then, the field

$$u_n^{ex}(y) := [E_n(y, z)]_1, \quad y \in \mathbb{R}^2 \setminus D,$$

solves the stationary problem.

Hyperbolic case 1: numerical results

M	$(\tilde{u}_{0,M})_1(y)$	$(\tilde{u}_{1,M})_1(y)$	$(\tilde{u}_{2,M})_1(y)$
8	0.293581559232	-0.084483725080	-0.146079079028
16	0.284988364785	-0.092525310787	-0.155666923858
32	0.285503199323	-0.092138738384	-0.155404881866
64	0.285503741272	-0.092138337605	-0.155404627504
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Table: The first component of the computed and exact (blue) solutions for $\kappa = 1$, at the measurement point $y = (1.5, 1)$.

Hyperbolic case 1: numerical results

M	$(\tilde{u}_{0,M})_2(y)$	$(\tilde{u}_{1,M})_2(y)$	$(\tilde{u}_{2,M})_2(y)$
8	0.081036497084	0.028667287783	-0.011013685642
16	0.071649152048	0.017647071803	-0.021814816353
32	0.071756738147	0.017837149908	-0.021648258690
64	0.071756880072	0.017837482038	-0.021647898034
	0.071756880072	0.017837482039	-0.021647898033

Table: The second component of the computed and exact (blue) solutions for $\kappa = 1$, at the measurement point $y = (1.5, 1)$.

Hyperbolic case 1: numerical results

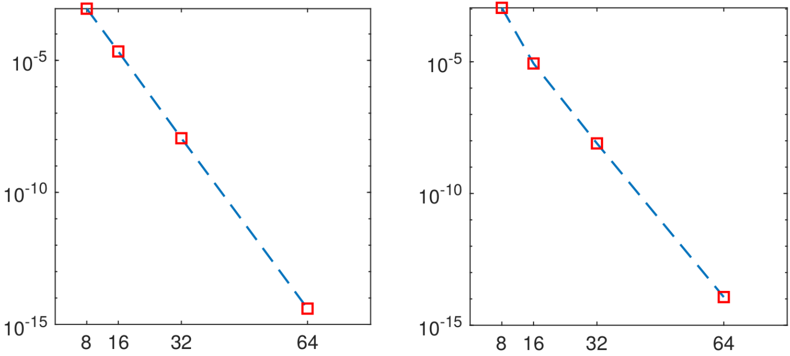

Figure: The L^2-norm of the difference between the computed and the exact solutions, in semi-logarithmic scale.

Hyperbolic case 1: numerical results

Example 2: We use the setup of the 1st example. We consider the spatial independent boundary function

$$f(x,t) = f(t)\begin{pmatrix}1\\1\end{pmatrix}, \quad \text{for} \quad f(t) = \frac{t^2}{4}e^{-t+2},$$

which admits the expansion

$$f(t) = \frac{\kappa e}{4}\sum_{n=0}^{\infty}\frac{2+\kappa n(\kappa(n-1)-4)}{(\kappa+1)^{n+3}}L_n(\kappa t).$$

Hyperbolic case 1: numerical results

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$$f(x,t) = f(t)\begin{pmatrix}1\\1\end{pmatrix}, \quad \text{for} \quad f(t) = \frac{t^2}{4}e^{-t+2},$$

which admits the expansion

$$f(t) = \frac{\kappa e}{4}\sum_{n=0}^{\infty}\frac{2+\kappa n(\kappa(n-1)-4)}{(\kappa+1)^{n+3}}L_n(\kappa t).$$

Even though the exact solution is unknown, we observe the convergence with respect to M, and N.

Hyperbolic case 1: numerical results

t	M	$N = 10$	$N = 15$	$N = 20$
1	16	0.104385591001	0.089274523790	0.089029847496
	32	0.104414411445	0.089311568889	0.089061508274
	64	0.104414399045	0.089311556699	0.089061496285
2	16	0.242963484605	0.268615650361	0.270356959200
	32	0.242960849558	0.268601030910	0.270336784091
	64	0.242963476811	0.268601022577	0.270336775642
3	16	0.294596074694	0.301687858830	0.299978778147
	32	0.294584218004	0.301675713615	0.299984785021
	64	0.294584215781	0.301675711488	0.299984782636

Table: The second component of the numerical solution for $\kappa = 1/2$, at the position $y = (0.5, -1.5)$.

Hyperbolic case 2: the inverse problem

Let $D \subset \mathbb{R}^2$ be doubly connected with two closed boundary curves Γ_1 and Γ_2.

We consider the Cauchy problem:

$$\begin{aligned} \frac{\partial^2 u}{\partial t^2} &= \Delta^* u, \quad && x \in D,\ t > 0, \\ \frac{\partial u}{\partial t}(\cdot, 0) = u(\cdot, 0) &= 0, && x \in D, \\ u = f_2, \quad Tu &= g_2, && x \in \Gamma_2, \end{aligned} \tag{3}$$

where f_2 and g_2 are given functions and T is the traction operator

$$Tv = \lambda \operatorname{div} v\, \nu + 2\mu\, (\nu \cdot \operatorname{grad})\, v + \mu \operatorname{div}(Qv)\, Q\nu.$$

with the outward unit normal vector ν to the boundary of D.

Hyperbolic case 2: the inverse problem

Let $D \subset \mathbb{R}^2$ be doubly connected with two closed boundary curves Γ_1 and Γ_2.

We consider the Cauchy problem:

$$\begin{aligned} \frac{\partial^2 u}{\partial t^2} &= \Delta^* u, \quad && x \in D,\ t > 0, \\ \frac{\partial u}{\partial t}(\cdot, 0) = u(\cdot, 0) &= 0, && x \in D, \\ u = f_2, \quad Tu &= g_2, && x \in \Gamma_2, \end{aligned} \tag{3}$$

where f_2 and g_2 are given functions and T is the traction operator

$$Tv = \lambda \operatorname{div} v\, \nu + 2\mu\, (\nu \cdot \operatorname{grad})\, v + \mu \operatorname{div}(Qv)\, Q\nu.$$

with the outward unit normal vector ν to the boundary of D.

The direct problem (given u on Γ_1) is well-posed [R.J. Knops and L.E. Payne, 1971].

Hyperbolic case 2: the inverse problem

Inverse Problem

Reconstruct the data on Γ_1, *given the system of equations* (3).

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We use the scaled Fourier expansion for u to obtain

$$\begin{aligned} \Delta^* u_n - \kappa^2 u_n &= \sum_{m=0}^{n-1} \beta_{n-m} u_m, && x \in D, \\ u_n = f_{2,n}, \quad T u_n &= g_{2,n}, && x \in \Gamma_2, \end{aligned}$$

where $n \in \mathbb{N}_0$, and $\beta_n = \kappa^2 (n+1)$.

Hyperbolic case 2: boundary integral equations

We construct

$$u_n(x) = \frac{1}{2\pi} \sum_{\ell=1}^{2} \sum_{m=0}^{n} \int_{\Gamma_\ell} E_{n-m}(x,y) q_m^\ell(y)\, ds(y), \quad x \in D,$$

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and as $u_n \to \Gamma_2$, using the jump properties, we obtain

$$\frac{1}{2\pi} \sum_{\ell=1}^{2} \int_{\Gamma_\ell} E_0(x,y) q_n^\ell(y)\, ds(y) = F_n(x), \quad x \in \Gamma_2,$$

$$\frac{1}{2} q_n^2(x) + \frac{1}{2\pi} \sum_{\ell=1}^{2} \int_{\Gamma_\ell} T_x E_0(x,y) q_n^\ell(y)\, ds(y) = G_n(x), \quad x \in \Gamma_2,$$

for $n = 0, \ldots, N$, with the right-hand sides

Hyperbolic case 2: boundary integral equations

$$F_n(x) = f_{2,n}(x) - \frac{1}{2\pi}\sum_{\ell=1}^{2}\sum_{m=0}^{n-1}\int_{\Gamma_\ell} E_{n-m}(x,y)q_m^\ell(y)\,ds(y)$$

and

$$G_n(x) = g_{2,n}(x) - \frac{1}{2}\sum_{m=0}^{n-1} q_m^2(x) - \frac{1}{2\pi}\sum_{\ell=1}^{2}\sum_{m=0}^{n-1}\int_{\Gamma_\ell} T_x E_{n-m}(x,y)q_m^\ell(y)\,ds(y).$$

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- As before, the kernels E_n contain logarithmic singularities but $T_x E_n$ has strong singularity (Cauchy type).

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- As before, the kernels E_n contain logarithmic singularities but $T_x E_n$ has strong singularity (Cauchy type).
- The forward operator is injective and has dense range.

Hyperbolic case 2: quadrature rules

Compared to

$$E_n(x, y) = \ln|x-y| M^{n,1}(|x-y|) + M^{n,2}(|x-y|),$$

the traction admits the decomposition

$$T_x E_n(x, y) = \ln|x-y| W^{n,1}(|x-y|) + \frac{1}{|x-y|} W^{n,1}(|x-y|),$$

for some analytic matrix-valued functions $M^{n,k},\ W^{n,k},\ k = 1, 2.$

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for some analytic matrix-valued functions $M^{n,k},\ W^{n,k},\ k = 1,2.$

Given the parametrization, we approximate

$$\begin{aligned} T_x E_n(x_k(s), x_\ell(\sigma)) &= \ln\left(\frac{4}{e}\sin^2\frac{s-\sigma}{2}\right) Q^{n,1}_{\ell,\ell}(s,\sigma) \\ &\quad + \cot\frac{\sigma - s}{2} Q^{n,2}_{\ell,\ell}(s) + Q^{n,3}_{\ell,\ell}(s,\sigma) \end{aligned}$$

for $s \neq \sigma$, $k,\ell = 1,2$, $n = 0,\dots,N$.

Hyperbolic case 2: numerical results

Example 3: We set

$$f_{2,n}(x) = [E_n(x, z_1)]_1, \quad g_{2,n}(x) = [TE_n(x, z_1)]_1, \quad x \in \Gamma_2.$$

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The source and the measurement points are

$$z_1 = (3, 3), \quad \text{and} \quad z_2 = \frac{\sqrt{2}}{2}(1, 1) \in \Gamma_1$$

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We consider noisy data of the form

$$g^{\delta}_{2,n} = g_{2,n} + \delta \frac{\|g_{2,n}\|_2}{\|v\|_2} v,$$

for a normally distributed random variable v.

Hyperbolic case 2: numerical results

κ	M	$f_{1,0}(z_2)$	$f_{1,5}(z_2)$	$f_{1,10}(z_2)$
0.5	16	0.160838025793	0.028252624954	0.008865680398
	32	0.160981797423	0.028078500923	0.008781179908
		0.160981796003	0.028078500985	0.008781181106
1	16	0.035987791353	0.008580531214	-0.037640150526
	32	0.036002107356	0.008720257700	0.002279598771
		0.036002151515	0.008720380239	0.002279504879

Table: The reconstructed $f_{1,n}$ on Γ_1. The regularization parameter is 10^{-10} for $\kappa = 0.5$, and 10^{-8} for $\kappa = 1$.

Hyperbolic case 2: numerical results

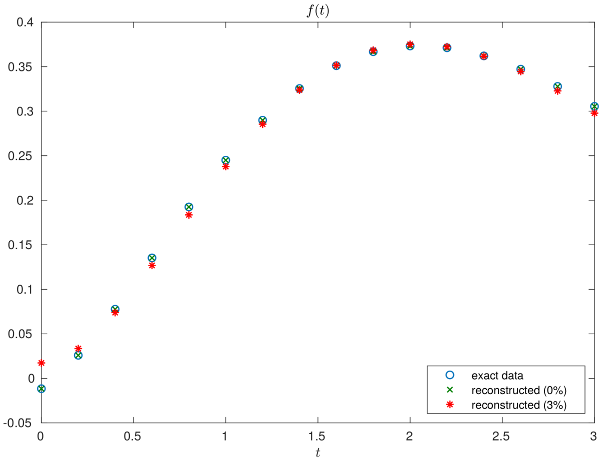

Figure: The reconstructed function $f_1(x,t)$ on Γ_1 for $N = 10$ and $M = 64$.

Hyperbolic case 2: numerical results

We define

$$e_f^2(n) = \frac{\int_{\Gamma_1} \left(f_{1,n} - [E_n(\cdot, z_1)]_1\right)^2 dx}{\int_{\Gamma_1} ([E_n(\cdot, z_1)]_1)^2 \, dx}, \quad e_g^2(n) = \frac{\int_{\Gamma_1} \left(g_{1,n} - [TE_n(\cdot, z_1)]_1\right)^2 dx}{\int_{\Gamma_1} ([TE_n(\cdot, z_1)]_1)^2 \, dx}$$

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	$\delta = 0\%$		$\delta = 3\%$	
n	$e_f(n)$	$e_g(n)$	$e_f(n)$	$e_g(n)$
5	5.9908E−06	7.7404E−05	0.14558	0.71139
10	1.0238E−05	8.0608E−05	0.14038	0.65011
15	1.0642E−05	4.6722E−05	0.39071	0.64990
20	7.8734E−05	5.5732E−04	0.41676	0.82314

Table: Relative errors for the source point $z_1 = (3,\ 3)$, and $M = 32$.

Hyperbolic case 2: numerical results

We compare $\tilde{E}(t)$ with

$$\tilde{u}(x,t) = \kappa \sum_{n=0}^{N-1} u_n(x;M) L_n(\kappa t), \quad x \in D.$$

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		$N = 15$		$N = 20$	
t	M	$\delta = 0\%$	$\delta = 3\%$	$\delta = 0\%$	$\delta = 3\%$
1	16	0.46387720	0.56160816	0.34605948	0.54725934
	32	0.54302224	0.54522283	0.54096499	0.53709220
	$[\tilde{E}(t)]_1$	0.543055422		0.540998187	
2	16	0.59385442	0.46302021	-1.41326137	0.36623652
	32	0.48564171	0.48223841	0.47964119	0.47575211
	$[\tilde{E}(t)]_1$	0.485671892		0.479683952	

Hyperbolic case 3: the inverse problem

Let D be doubly-connected with smooth boundary $\Gamma = \Gamma_1 \cup \Gamma_2$.

We consider the initial BVP:

$$\begin{aligned}
\frac{1}{\alpha^2}\frac{\partial^2 u}{\partial t^2}(x,t) - \Delta u(x,t) &= 0, && x \in D,\ t > 0,\\
u(x,0) = \frac{\partial u}{\partial t}(x,0) &= 0, && x \in D,\\
u(x,t) &= 0, && x \in \Gamma_1,\ t \geq 0,\\
\frac{\partial u}{\partial \nu}(x,t) &= g(x,t), && x \in \Gamma_2,\ t \geq 0.
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Inverse Problem

Reconstruct Γ_1 from the knowledge of g and $u = f$ on Γ_2.

Hyperbolic case 3: time discretization

We consider the expansion

$$u(x,t) = \kappa \sum_{n=0}^{\infty} u_n(x) L_n(\kappa t),$$

and we obtain

$$\begin{aligned} \Delta u_n - \beta_0 u_n &= \sum_{m=0}^{n-1} \beta_{n-m} u_m, && x \in D, \\ u_n(x) &= 0, && x \in \Gamma_1, \\ u_n(x) = f_n(x), \quad \frac{\partial u_n}{\partial \nu} &= g_n(x), && x \in \Gamma_2, \end{aligned} \tag{5}$$

for $n \in \mathbb{N}_0$, where $\beta_n = (n+1)\kappa^2/\alpha^2$.

Hyperbolic case 3: boundary integral equations

We use

$$u_n(x) = \frac{1}{\pi} \sum_{\ell=1}^{2} \sum_{m=0}^{n} \int_{\Gamma_\ell} E_{n-m}(x, y)\, \phi_m^\ell(y) ds(y), \quad x \in D$$

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and we obtain

$$\frac{1}{\pi} \sum_{\ell=1}^{2} \int_{\Gamma_\ell} E_0(x,y) \phi_n^\ell(y) ds(y) = F_{1,n}(x), \quad x \in \Gamma_1,$$

$$\phi_n^2(x) + \frac{1}{\pi} \sum_{\ell=1}^{2} \int_{\Gamma_\ell} \frac{\partial E_0}{\partial n(x)}(x,y) \phi_n^\ell(y) ds(y) = G_n(x), \quad x \in \Gamma_2,$$

$$\frac{1}{\pi} \sum_{\ell=1}^{2} \int_{\Gamma_\ell} E_0(x,y) \phi_n^\ell(y) ds(y) = F_{2,n}(x), \quad x \in \Gamma_2,$$

Hyperbolic case 3: boundary integral equations

for the right-hand side

$$F_{1,n}(x) = -\frac{1}{\pi}\sum_{\ell=1}^{2}\sum_{m=0}^{n-1}\int_{\Gamma_\ell} E_{n-m}(x,y)\phi_m^\ell(y)ds(y),$$

$$G_n(x) = g_n(x) - \sum_{m=0}^{n-1}\phi_m^2(x) - \frac{1}{\pi}\sum_{\ell=1}^{2}\sum_{m=0}^{n-1}\int_{\Gamma_\ell} \frac{\partial E_{n-m}}{\partial n(x)}(x,y)\phi_m^\ell(y)ds(y),$$

$$F_{2,n}(x) = f_n(x) - \frac{1}{\pi}\sum_{\ell=1}^{2}\sum_{m=0}^{n-1}\int_{\Gamma_\ell} E_{n-m}(x,y)\phi_m^\ell(y)ds(y).$$

Hyperbolic case 3: boundary integral equations

Iterative scheme:

1. Given an initial guess for Γ_1, we solve the first two parametrized equations for the densities.

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Comments

- The Fréchet derivative operator is injective.

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Comments

- The Fréchet derivative operator is injective.
- We obtain an overdetermined system of equations for the perturbed boundary.

Hyperbolic case 3: numerical implementation

We consider

$$x_1(s) = \{r(s)(\cos s, \sin s) : \ s \in [0, 2\pi]\},$$

and we apply trigonometric interpolation

$$q(s) \approx \sum_{j=0}^{2J} q_j \tau_j(s), \quad \mathbb{N} \ni J \ll M,$$

with

$$\tau_j(s) = \begin{cases} \cos(js), & \text{for } j = 0, ..., J, \\ \sin((j-J)s), & \text{for } j = J+1, ..., 2J. \end{cases}$$

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We solve the linear system with Tikhonov regularization

$$\min_{\boldsymbol{q}} \{\|\boldsymbol{A}\boldsymbol{q} - \boldsymbol{b}\|_2^2 + \lambda \|\boldsymbol{q}\|_2^2\},$$

for a regularization parameter $\lambda > 0$.

Hyperbolic case 3: numerical results

Example 4: We set $M = 64$, $\kappa = 1$, $\alpha = 1$ and $N = 10$. The results are presented for $J = 13$:

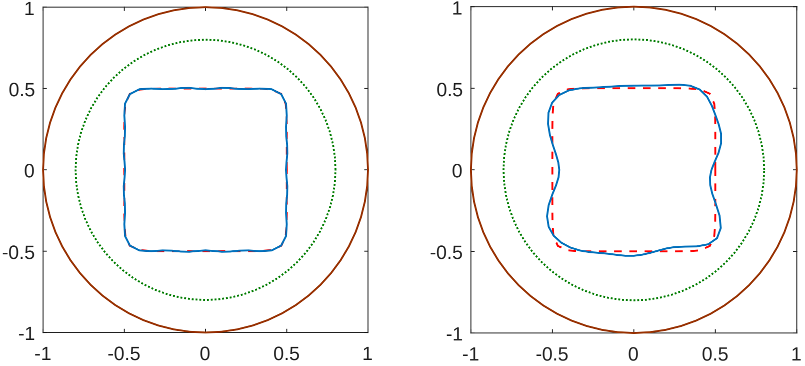

Figure: Reconstructions of Γ_1 for exact data (left) and data with 3% noise (right).

Hyperbolic case 3: numerical results

Example 5: The results are presented for $J = 5$:

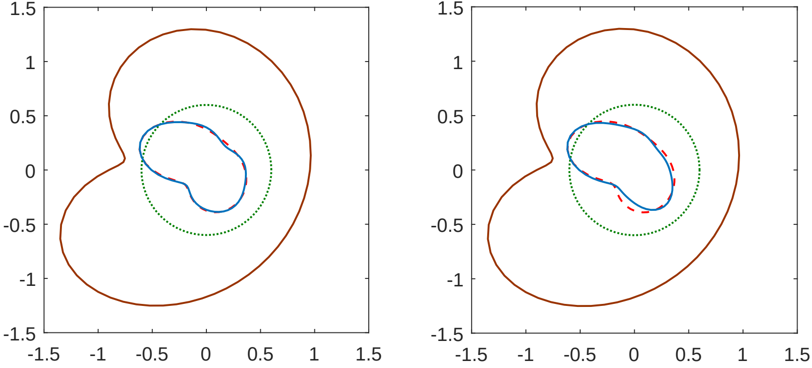

Figure: Reconstructions of Γ_1 for exact data (left) and noisy data (right).

Parabolic case: the inverse problem

We consider the Cauchy problem:

$$
\begin{aligned}
\frac{1}{\alpha}\frac{\partial u}{\partial t}(x,t) - \Delta u(x,t) &= 0, && x \in D,\ t > 0,\\
u(x,0) &= 0, && x \in D,\\
u(x,t) &= 0, && x \in \Gamma_1,\ t \geq 0,\\
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The direct problem is well-posed [A. Friedman, 1964].

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The direct problem is well-posed [A. Friedman, 1964].

Inverse Problem

Reconstruct Γ_1 *from the knowledge of the thermal flux* g *and* $u = f$ *on* Γ_2.

Parabolic case: time discretization

Given the scaled Fourier expansion of u, we derive

$$
\begin{aligned}
\Delta u_n - \beta^2 u_n &= \beta^2 \sum_{m=0}^{n-1} u_m, && x \in D,\\
u_n(x) &= 0, && x \in \Gamma_1,\\
u_n(x) = f_n(x), \quad \frac{\partial u_n}{\partial \nu} &= g_n(x), && x \in \Gamma_2,
\end{aligned}
\tag{7}
$$

for $n \in \mathbb{N}_0$, where $\beta^2 = \kappa/\alpha$.

Parabolic case: boundary integral equations

We use

$$u_n(x) = \frac{1}{\pi} \sum_{\ell=1}^{2} \sum_{m=0}^{n} \int_{\Gamma_\ell} E_{n-m}(x, y)\, \phi_m^\ell(y) ds(y), \quad x \in D$$

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$$\frac{1}{\pi} \sum_{\ell=1}^{2} \int_{\Gamma_\ell} E_0(x,y) \phi_n^\ell(y) ds(y) = F_{2,n}(x), \quad x \in \Gamma_2,$$

Parabolic case: numerical results

Example 6: We set $M = 64$, $\kappa = 1$ and $N = 10$.

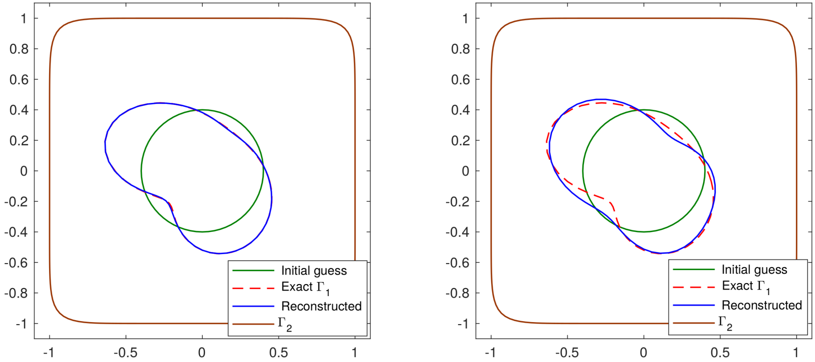

Figure: Reconstructions of Γ_1, using $J = 5$, for exact data (left) and data with 3% noise (right).

Parabolic case: numerical results

Example 7: Here, $J = 7$ and we use $r_0 = 0.5$:

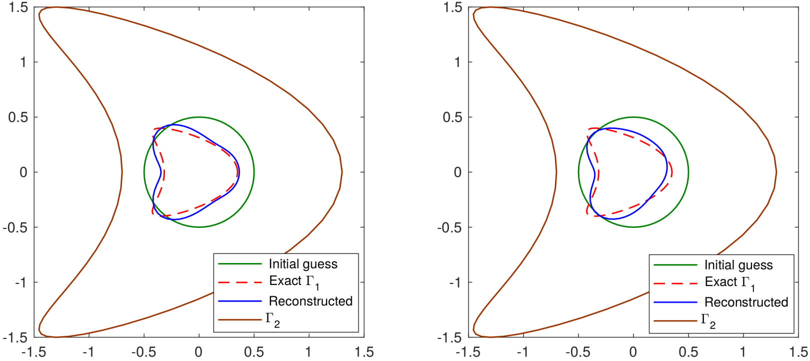

Figure: Reconstructions of Γ_1 for exact data (left) and noisy data (right).

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