High-contrast high-resolution
Hybrid Inverse Problems

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Outline

1. Generalities about Coupled-Physics (Hybrid) Inverse Problems

2. Photo-acoustic Tomography

3. Elastography

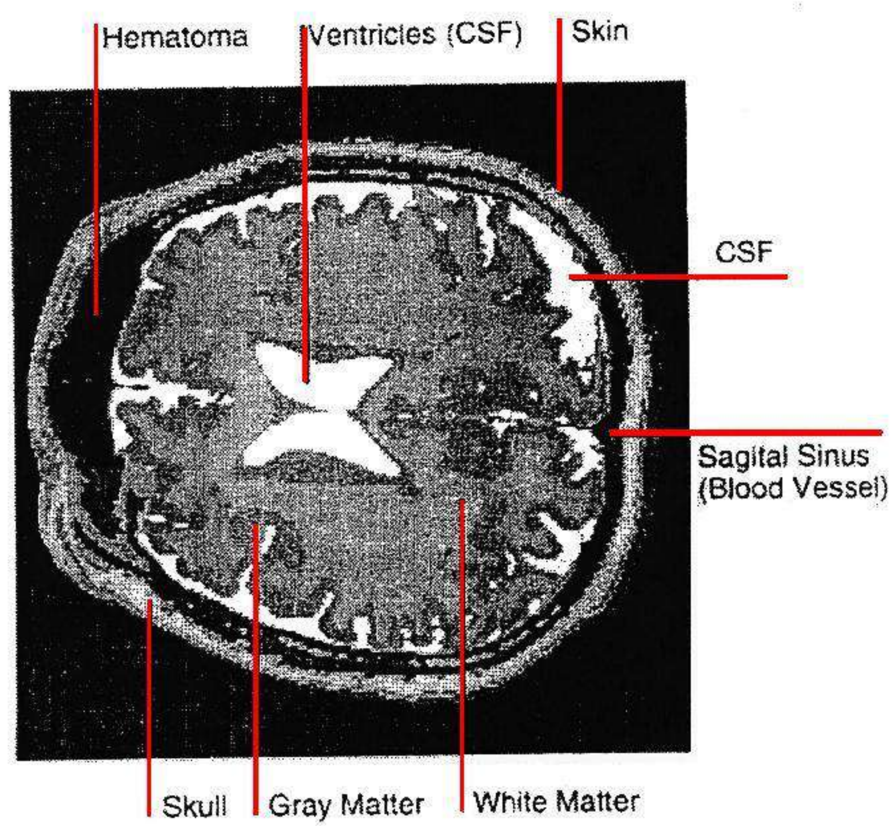
4. Other HIP & Elliptic Theory

5. HIP with Large Redundancies

6. Qualitative properties, CGOs, Runge approximation

`https://www.stat.uchicago.edu/~guillaumebal/PAPERS/IntroductionInverseProblems.pdf`

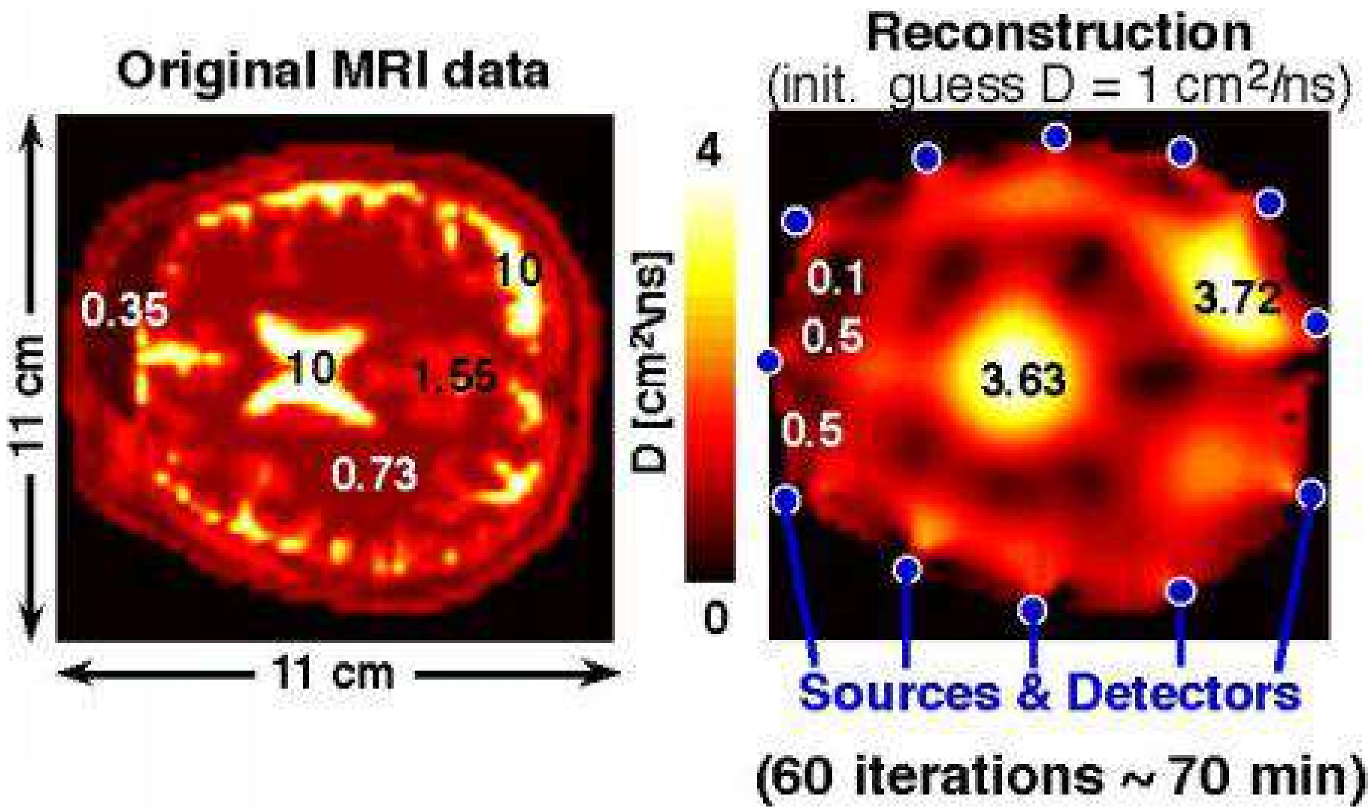
Optical Constrast in medical imaging



Segmented MRI data for a human brain.

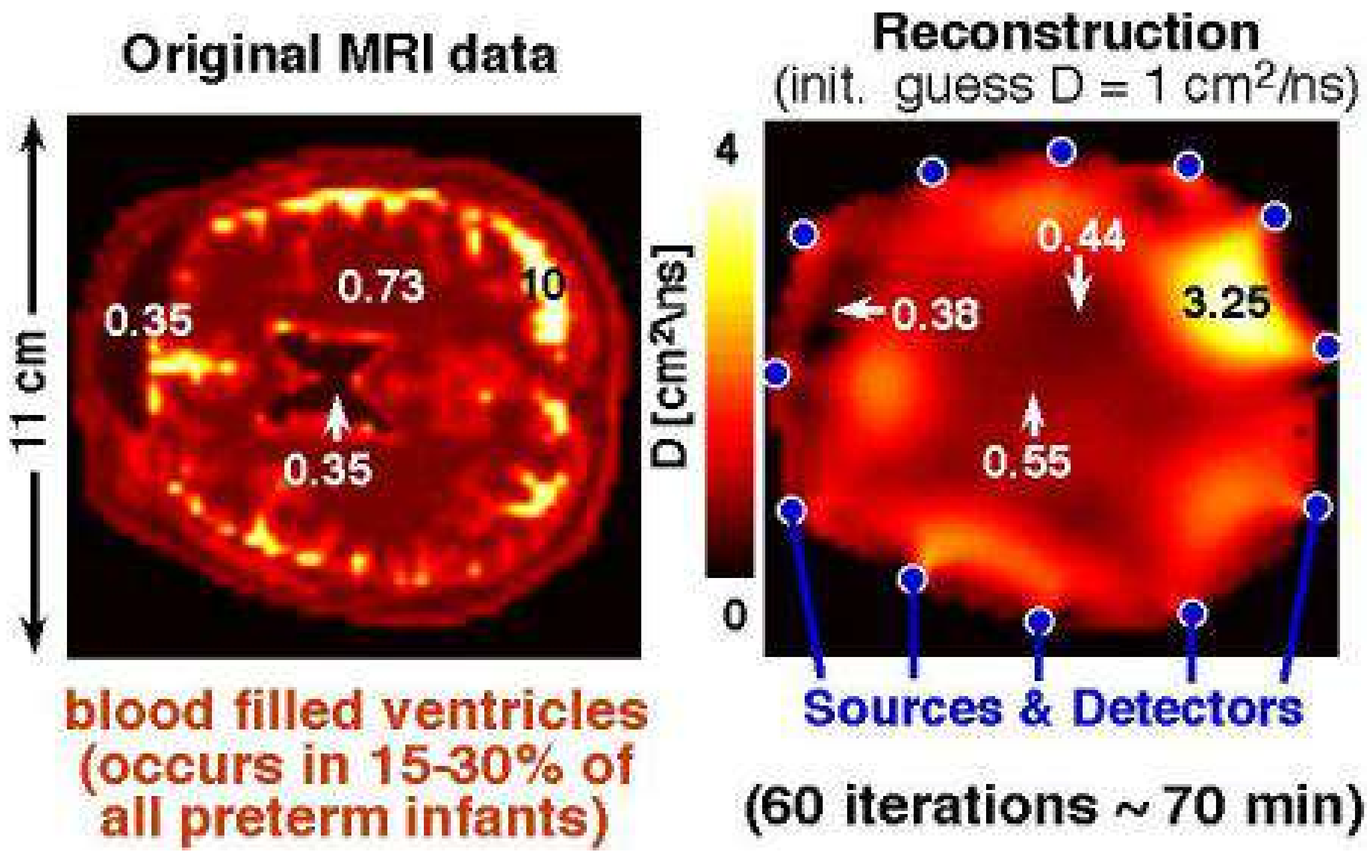
Anatomy of a human brain based on MRI data.

Optical properties of a healthy brain



Brain with clear ventricle in neonate. *(A.H.Hielscher, Columbia biomed.)*

Scattering for blood-filled ventricle



Brain with blood-filled ventricle in neonate. *(A.H. Hielscher, Columbia biomed.)*

Mathematical model: Calderón problem

Optical Tomography (neglecting absorption to simplify) is modeled by:

$$-\nabla\cdot\gamma(x)\nabla u = 0 \quad \text{in} \quad X \qquad \text{and} \qquad u = f \quad \text{on} \quad \partial X.$$

Calderón problem: Reconstruction of $\gamma(x)$ from knowledge of the Dirichlet-to-Neumann map Λ_γ, where $f \mapsto \Lambda_\gamma f = \gamma\nu\cdot\nabla u_{|\partial X}$ on the boundary ∂X.

The Calderón problem is **injective**: $\Lambda_\gamma = \Lambda_{\tilde\gamma} \Longrightarrow \gamma = \tilde\gamma$.

[Sylvester-Uhlmann 87, Nachman 88, Brown-Uhlmann 97, Astala-Päivärinta 06, Haberman-Tataru 11].

The Calderón problem is **unstable**: The modulus of continuity is logarithmic [Alessandrini 88], which results in **low resolution**:

$$\|\gamma-\tilde\gamma\|_{\mathfrak{X}} \le C\Big|\ln\|\Lambda_\gamma-\Lambda_{\tilde\gamma}\|_{\mathfrak{Y}}\Big|^{-\delta}.$$

Complex Geometrical Optics solutions

Injectivity of the Calderón problem is proved by showing that $q_1 = q_2$

when $(\Delta - q_i)u_i = 0$ and $\int_X (q_1 - q_2)\, u_1 u_2 \, dx = 0.$

Statement on the **density** of products of (almost-) harmonic solutions.

CGO solutions are of the form

$$\boxed{u_\rho = e^{\rho\cdot x}\Big(1 + \psi_\rho(x)\Big)} \qquad \rho = k + ik^\perp \in \mathbb{C}^n, \quad |k| = |k^\perp|,\ k\cdot k^\perp = 0.$$

Property: $|\rho||\psi_\rho|$ is bounded (ψ_ρ is small as $|\rho|\to\infty$).
Choosing ρ_1 and ρ_2 such that $\rho_1 + \rho_2 = i\xi \in \mathbb{R}^n$ and $|\rho_1|, |\rho_2| \to \infty$:

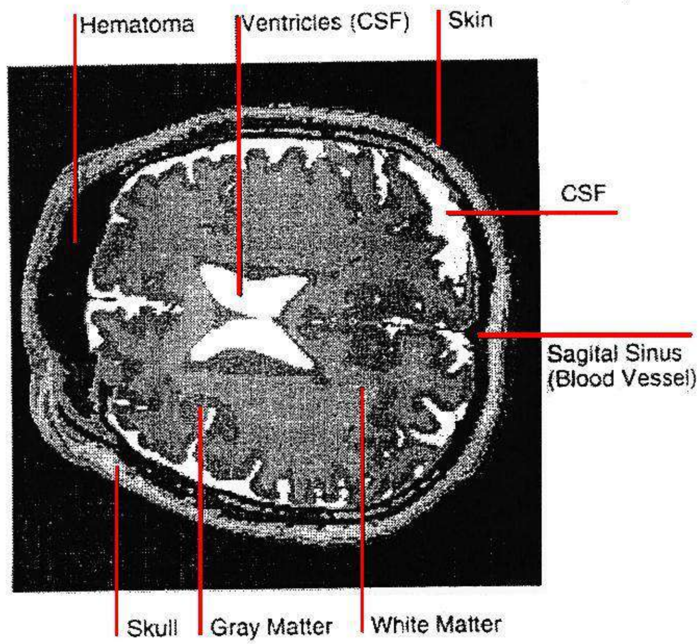
$$\lim_{|\rho_1|,|\rho_2|\to\infty} \int_X (q_1 - q_2) u_{\rho_1} u_{\rho_2} dx = \int_X (q_1 - q_2) e^{i\xi\cdot x} dx = 0.$$

However, $|u_1|, |u_2| \sim e^{|\xi|}$ to *determine* $\hat{q}(\xi)$: the Calderón problem is a severely ill-posed inverse problem with **low resolution** capabilities.

High resolution Ultrasound

Ultrasound Imaging (ultrasonography) is an imaging modality that provides **high resolution**. However, it may display **low contrast** in soft tissues.

High resolution MRI



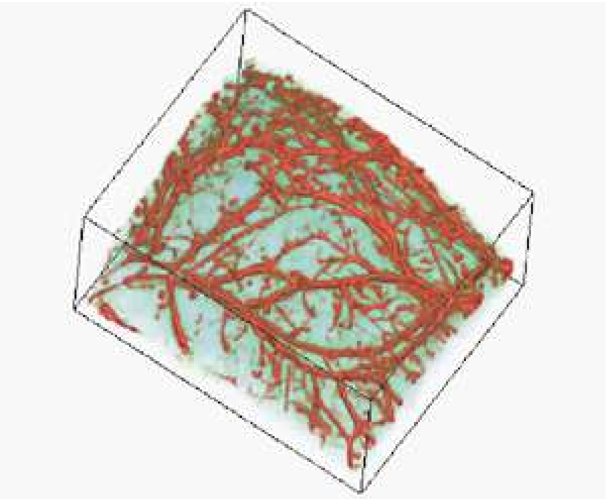
Segmented MRI data for a human brain.

MRI also provides **high resolution** and may also display **low contrast** in soft tissues.

High Contrast and High Resolution

High-contrast low-resolution modalities: OT, EIT, Elastography. Based on elliptic models that do not propagate singularities (well).

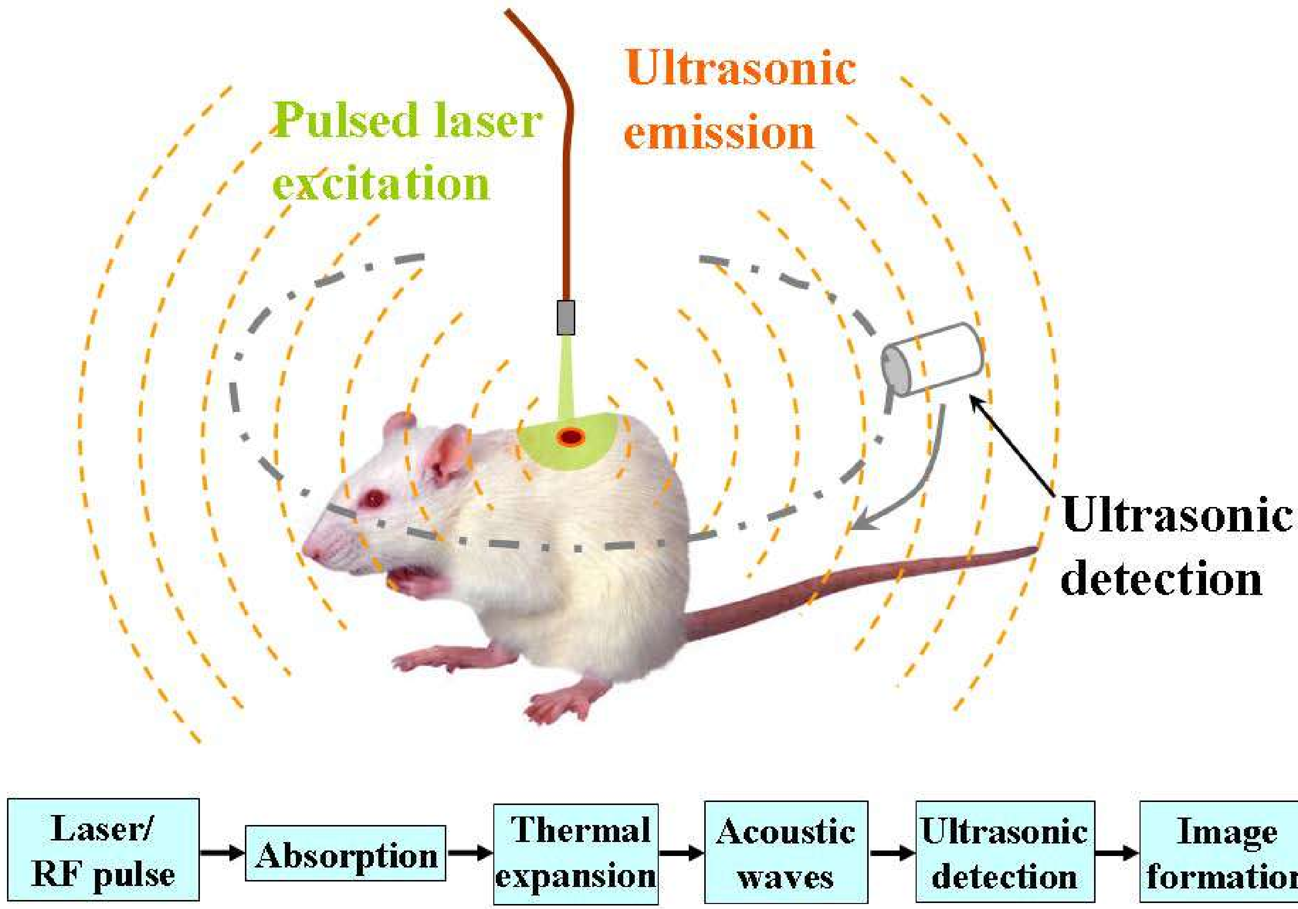
High-resolution low-contrast (soft tissues): M.R.I, Ultrasound, (X-ray CT). Singularities propagate: WF(data) determines WF(parameters).



High-contrast & High-resolution: Hybrid Inverse Problems (HIP): Physical Coupling between one modality in each category.

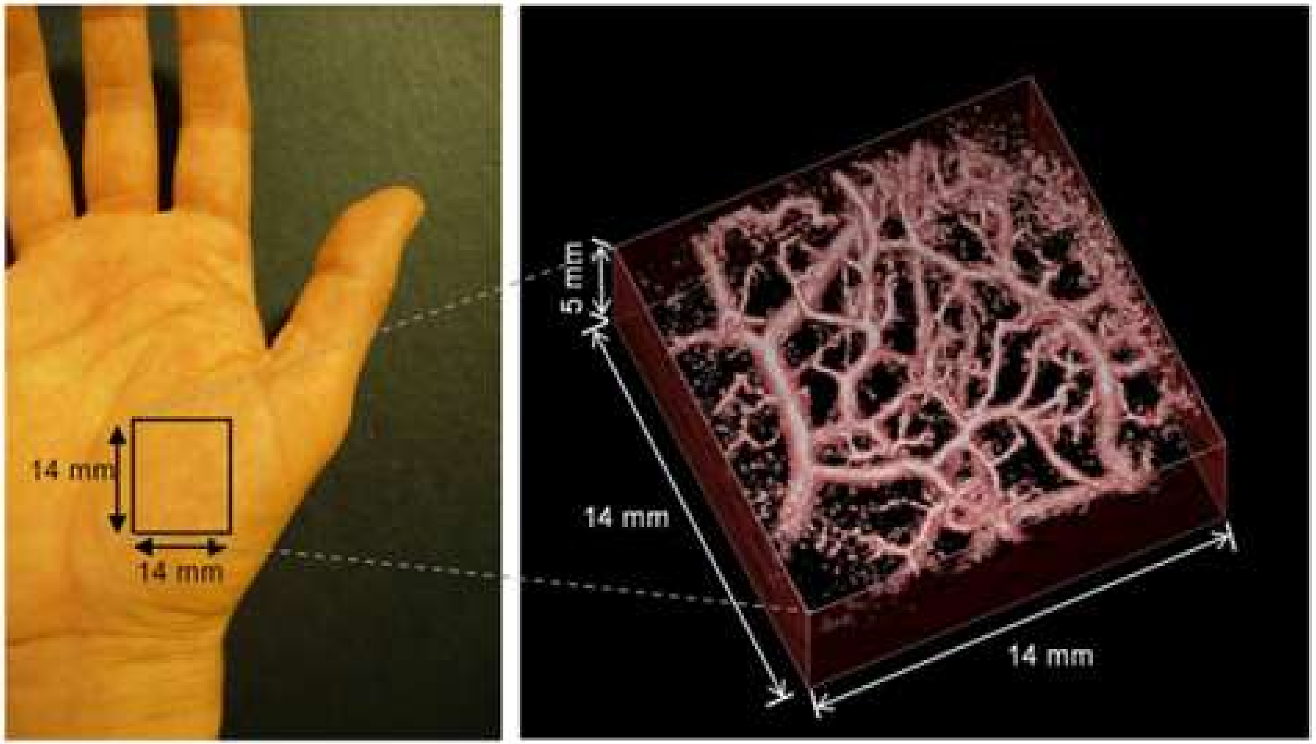
HIP are typically Low Signal.

The Photo-acoustics Effect



Coupling between (Near-Infra-Red) Radiation and Ultrasound.

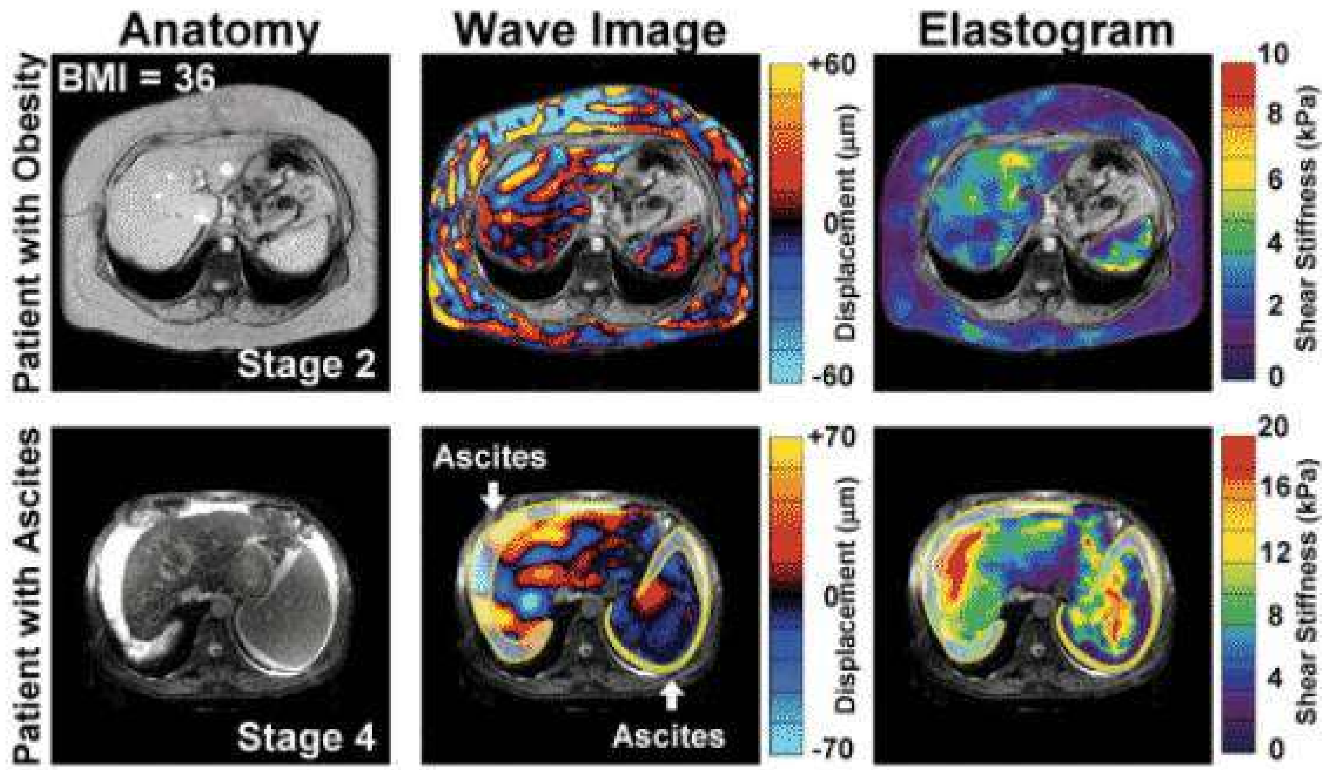
Experimental results in Photoacoustics



Reconstruction of Ultrasound generated by Photo-Acoustic effect.

From Paul Beard's Lab, University College London, UK.

Elastography and Magnetic Resonance

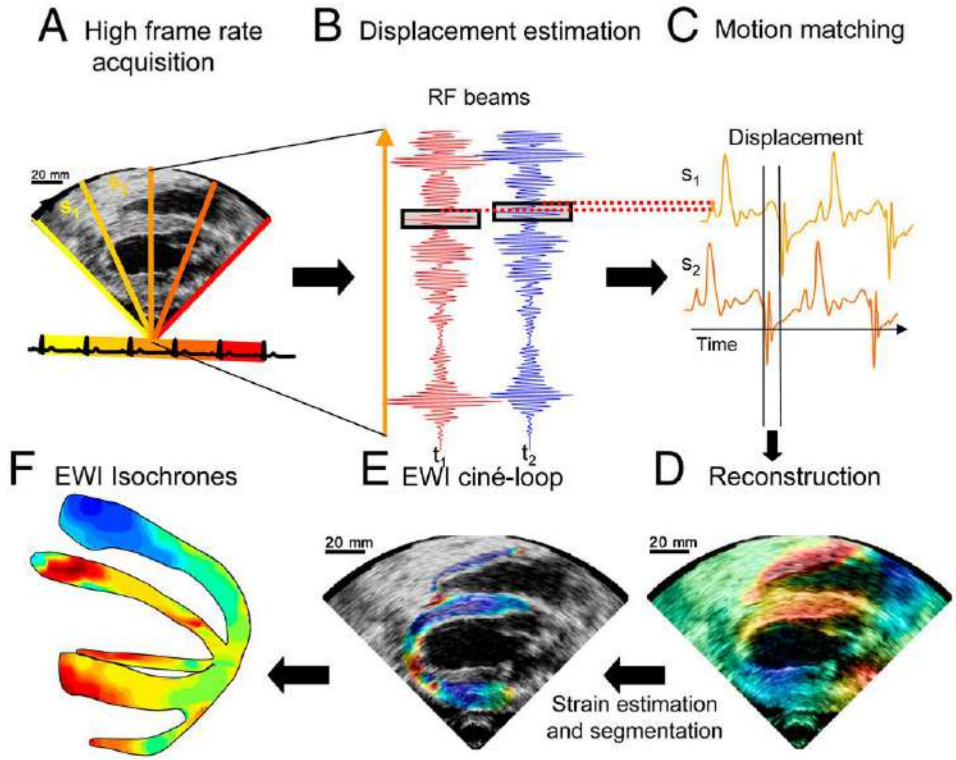


Assessment of Hepatic Fibrosis by Liver Stiffness

Coupling between Elastic Waves and Magnetic Resonance Imaging

From Richard L. Ehman's Lab (Mayo Clinic, Rochester, MN)

Elastography and Ultrasound



Electromechanical Wave Imaging (EWI) of the heart

Coupling between Transient Elastic Waves and Ultrasound

From Elisa Konofagou's Lab (Columbia University)

Hybrid inverse problems and internal functionals

• Hybrid (Multi-Physics) Inverse Problems (HIP) typically involve two-steps.

• The first step solves a **high resolution** inverse **boundary** problem, for instance by inverting Ultrasound Measurements or Magnetic Resonance Measurements.

• The *outcome* of the first step is the availability of **Internal Functionals** of the parameters of interest. HIP theory aims to address:

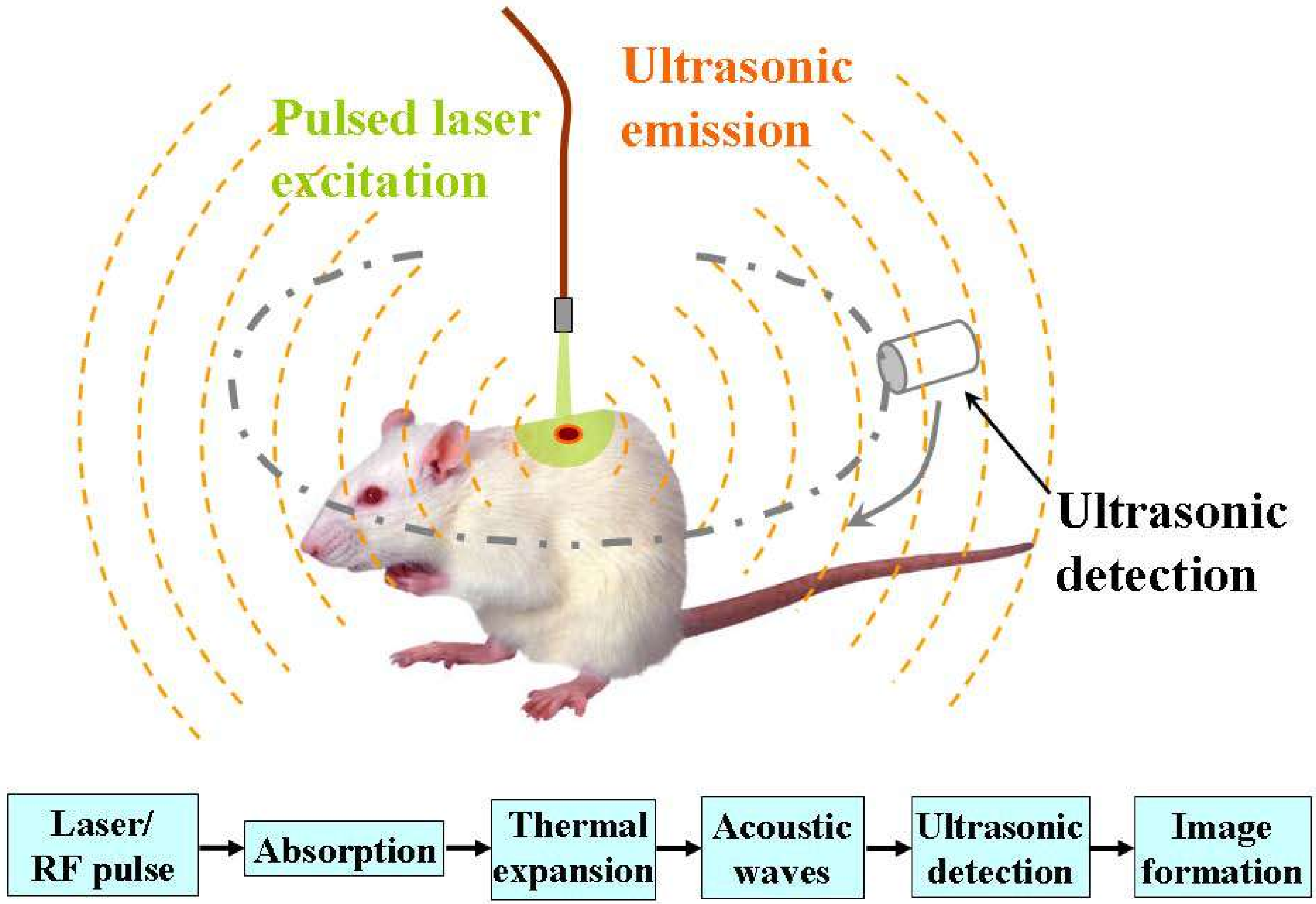
- Which parameters can be uniquely determined
- With which stability (resolution)
- Under which illumination (boundary probing) mechanism.

Photo-Acoustic Tomography

High Contrast: Optical (or Electromagnetic) properties

High Resolution : Ultrasound

The Photo-acoustics Effect



Coupling between (Near-Infra-Red) Radiation and Ultrasound.

Acoustic Modeling of PAT

Ultrasound propagation is modeled by:

$$\frac{1}{c_s^2}\frac{\partial^2 p}{\partial t^2} = \Delta p \text{ in } \mathbb{R}^+ \times \mathbb{R}^n; \quad \boxed{p(0,x) = \Gamma(x)\sigma(x)u(x),} \quad \partial_t p(0,x) = 0 \text{ in } \mathbb{R}^n,$$

with Γ the Grüneisen coefficient and σ the absorption coefficient.

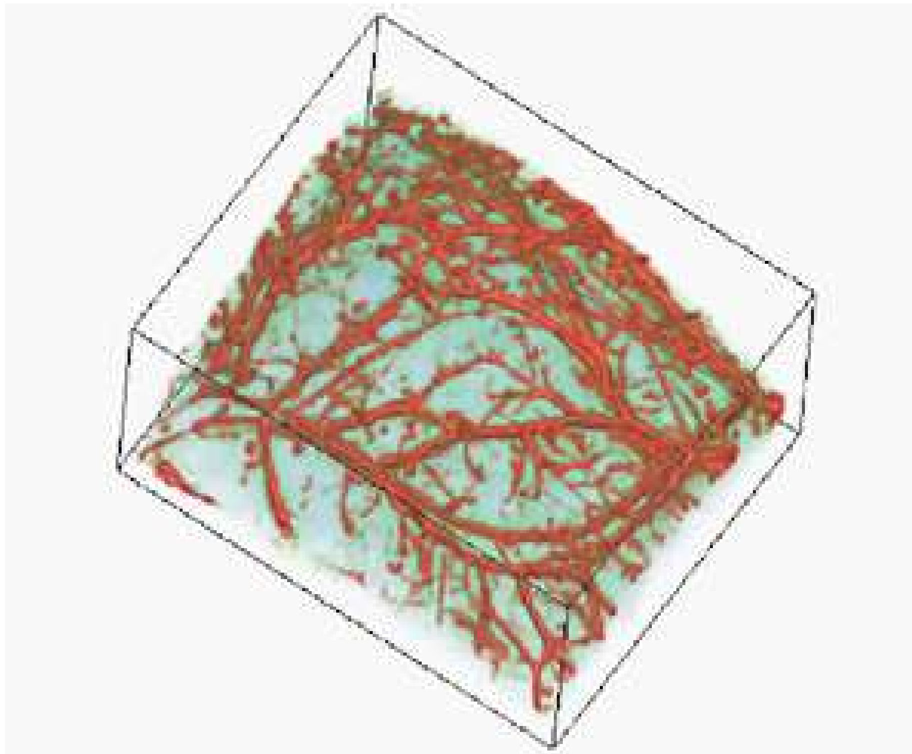
The PAT measurement operator (with γ additional optimal parameters):

$$\boxed{(\gamma(x), \sigma(x), \Gamma(x)) \mapsto \Big\{ p(t,x) | \, t > 0, x \in \partial X \Big\}}.$$

The *First Step* in PAT: reconstruct $p(0,x)$ from data. For $X = B(0,1)$:

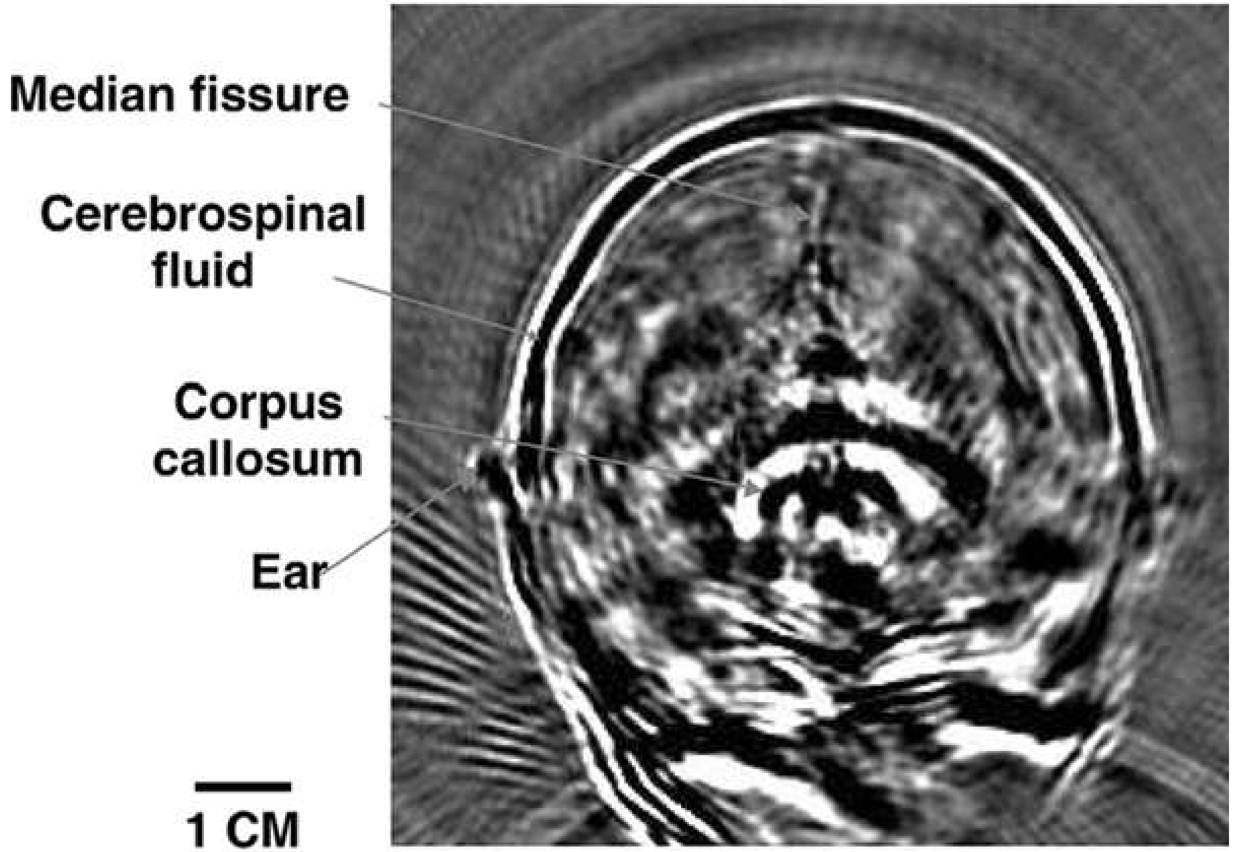
$$H(x) := p(0,x) = \frac{1}{8\pi^2}\nabla_x \cdot \int_{|y|=1} \nu(y) \Big(\frac{1}{t}\frac{\partial}{\partial t}\frac{p(t,y)}{t} \Big)_{t=|y-x|} dS_y \quad (c_s = 1).$$

Experimental results in Photoacoustics



Reconstruction of $H(x)$. *From Lihong Wang's Lab*

Extensive theoretical literature by Finch, Rakesh, Patch; Kuchment, Kunyansky, Hristova, Lin; Stefanov, Uhlmann (non-constant c_s); Scherzer et al.; Natterer.



Artifacts caused by resonant cavity (skull) showing some outstanding problems

Quantitative step of PAT: light modeling

(i) Light modeling as a boundary value radiative transfer problem:

$$v\cdot\nabla_x u+\sigma_t(x)u-\int_{\mathbb{S}^{n-1}}k(x,v',v)u(x,v')dv'=0,\qquad (x,v)\in X\times\mathbb{S}^{n-1}$$
$$u(x,v)=\phi(x,v)\qquad (x,v)\in\Gamma_-=\{(x,v)\in\partial X\times\mathbb{S}^{n-1},\quad v\cdot\nu(x)<0\},$$

for all **illuminations** ϕ and consider the data acquisition operator

$$\phi(x,v)\mapsto H(x):=\Gamma(x)\sigma(x)\int_{\mathbb{S}^{n-1}}u(x,v)dv;\qquad \sigma(x)=\sigma_t(x)-\int_{\mathbb{S}^{n-1}}k(x,v',v)dx'.$$

What is reconstructed in (σ_t,k) (Γ known): B. Jollivet Jugnon IP09; Ren 15.

(ii) Light modeling in diffusive regime: **optical radiation** is modeled by:

$$-\nabla\cdot\gamma(x)\nabla u_j+\sigma(x)u_j=0 \text{ in } X;\qquad u=f_j \text{ on } \partial X\qquad \textbf{Illumination,}$$

with a data acquisition operator $f_j(x)\mapsto H(x)=\Gamma(x)\sigma(x)u_j(x)$.

QPAT with two measurements (illuminations)

$$-\nabla\cdot\gamma(x)\nabla u_j+\sigma(x)u_j=0 \text{ in } X,\quad u_j=f_j \text{ on } \partial X;\qquad H_j(x)=\Gamma(x)\sigma(x)u_j(x).$$

Let (f_1,f_2) providing (H_1,H_2). Define $\beta=H_1^2\nabla\frac{H_2}{H_1}$. **IF:** $0\not\equiv\beta\in W^{1,\infty}(X)$:

Theorem[B.-Uhlmann 10, B.-Ren 11]
(i) (H_1,H_2) uniquely determine

$$\chi(x):=\frac{\sqrt{\gamma}}{\Gamma\sigma}(x),\qquad q(x):=-\Big(\frac{\Delta\sqrt{\gamma}}{\sqrt{\gamma}}+\frac{\sigma}{\gamma}\Big)(x).$$

(ii) (H_1,H_2) uniquely determine the *whole* data acquisition operator:

$$f\in H^{\frac12}(\partial X)\mapsto\mathcal{H}(f)=H\in H^1(X).$$

• **Two well-chosen measurements suffice to reconstruct (χ,q) and thus (γ,σ,Γ) up to transformations leaving (χ,q) invariant**.
• If Γ is known, then (γ,σ) is uniquely reconstructed.

Quantitative PAT, transport, and diffusion

The proof is based on the *elimination* of σ to get

$$-\nabla\cdot\chi^2\Big[H_1^2\nabla\frac{H}{H_1}\Big]=0 \text{ in } X,\qquad \chi \text{ known on } \partial X.$$

Then we verify that $$q:=-\Big(\frac{\Delta\sqrt{\gamma}}{\sqrt{\gamma}}+\frac{\sigma}{\gamma}\Big)(x)=-\frac{\Delta(\chi H_1)}{\chi H_1}.$$

The **IF** ($\beta\not\equiv 0$) implies that the **vector field** $\beta=H_1^2\nabla\frac{u_2}{u_1}\neq 0$ a.e. This is a qualitative statement on the absence of (too many) critical points of elliptic solutions.

Theorem [B.-Ren 11] When *one* coefficient in (γ,σ,Γ) is known, then the other two are **uniquely** determined by the two functionals (H_1,H_2).

Reconstructions for constant Γ

Theorem[B.-Ren 11] When *one* coefficient in (γ, σ, Γ) is known, then the other two are **uniquely** determined by the two measurements (H_1, H_2).

For instance, assuming Γ known, we first solve

$$-\nabla\cdot\Big(\chi^2\big[H_1^2\nabla\frac{H_2}{H_1}\big]\Big) = 0 \text{ in } X, \qquad \chi^2 = h_1 \text{ on } \partial X.$$

Then, with $q(x)$ as before, we solve the elliptic equation

$$(\Delta + q)\sqrt{\gamma} + \frac{\Gamma}{\chi} = 0 \text{ in } X, \qquad \sqrt{\gamma} = h_2 \text{ on } \partial X.$$

We thus need to solve a **transport equation** and an *elliptic equation*.

Stability of the reconstruction (Γ known)

• Case of 2 measurements: $H = (H_1, H_2)$. **IF** $|\beta| \geq c_0 > 0$, then [B. Uhlmann IP 10], we find that for $k \geq 3$:

$$\|(\gamma,\sigma) - (\tilde\gamma,\tilde\sigma)\|_{C^{k-1}(X)} \leq C\|H - \tilde H\|_{(C^{k+1}(X))^2}.$$

Using CGO solutions, $|\beta| \geq c_0 > 0$ for (f_1, f_2) in an open set.

We thus observe a **loss of two derivatives** (sub-elliptic estimate).

• Case of $n+1$ measurements: $H = (H_1, \ldots, H_{n+1})$. Under appropriate assumptions [B. Uhlmann IP 10, CPAM 13], we find for $k \geq 3$:

$$\|\gamma - \tilde\gamma\|_{C^k(X)} + \|\sigma - \tilde\sigma\|_{C^{k+1}(X)} \leq C\|H - \tilde H\|_{(C^{k+1}(X))^{n+1}}.$$

We thus observe a **loss of one derivative** for γ and **none** for σ.

Why is $n+1$ significantly better than 2 ?

$$-\nabla\cdot\gamma(x)\nabla u_j+\sigma(x)u_j=0 \text{ in } X,\quad u_j=f_j \text{ on } \partial X; \qquad H_j(x)=\Gamma(x)\sigma(x)u_j(x).$$

The *elimination* of σ provides the transport equation

$$-\nabla\cdot[\chi^2 H_1^2]\nabla\frac{H_j}{H_1}=0 \text{ in } X, \qquad 2\le j\le n+1.$$

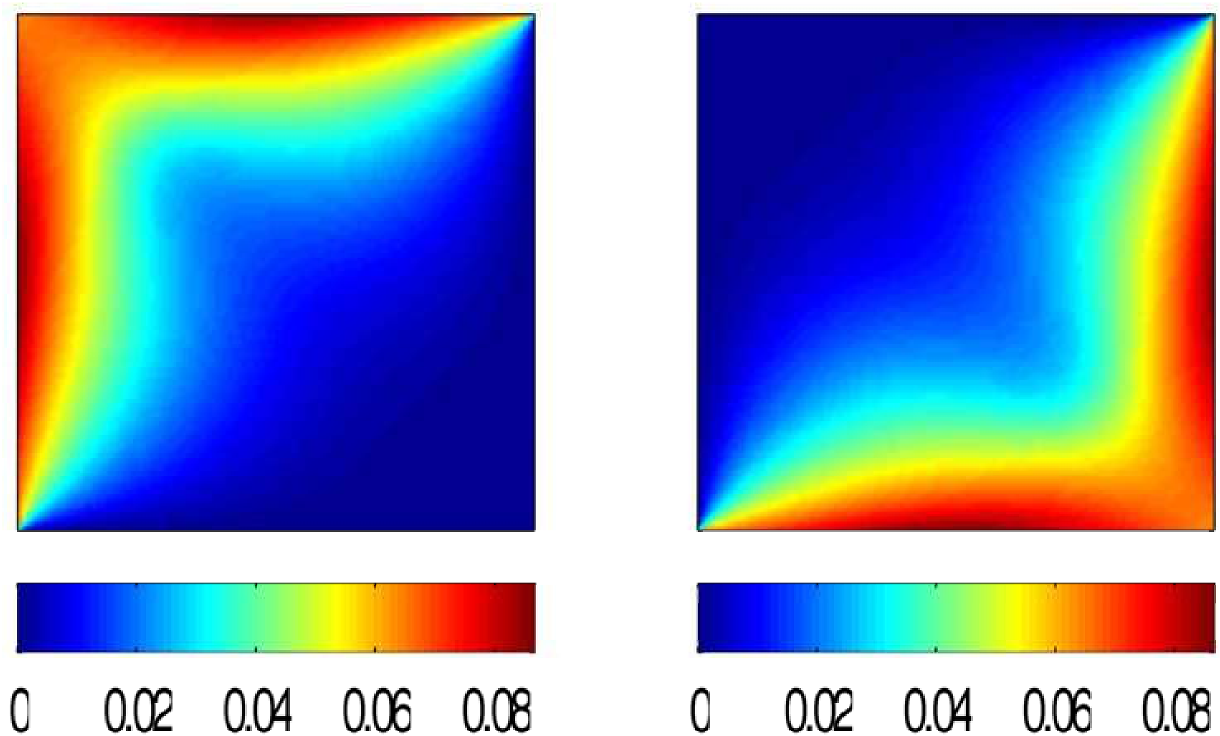
Let $\beta_j=\nabla\frac{H_j}{H_1}$ and $\zeta=\chi^2H_1^2$. We may recast the above equations as the over-determined **elliptic** system

$$\beta_j\cdot\nabla\zeta+(\nabla\cdot\beta_j)\zeta=0, \qquad \text{or} \qquad \nabla\zeta+\theta\zeta=0$$

if $\{\beta_j\}_{2\le j\le n+1}$ forms a basis of $\mathbb{R}^n$ at each point in X for a vector θ.

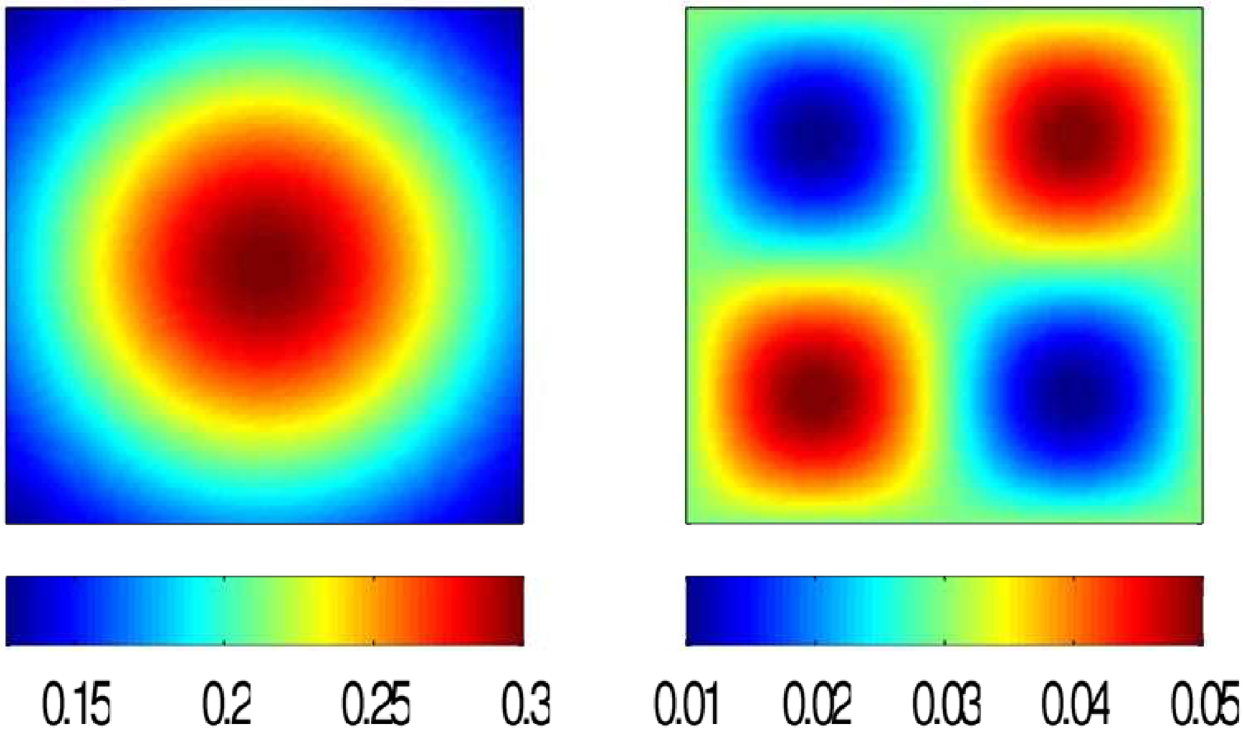
A redundant (and elliptic) system of transport equations enjoys better stability properties than a single transport equation.

Reconstructions in model $-\nabla \cdot \gamma \nabla u_j + \sigma u_j = 0$.



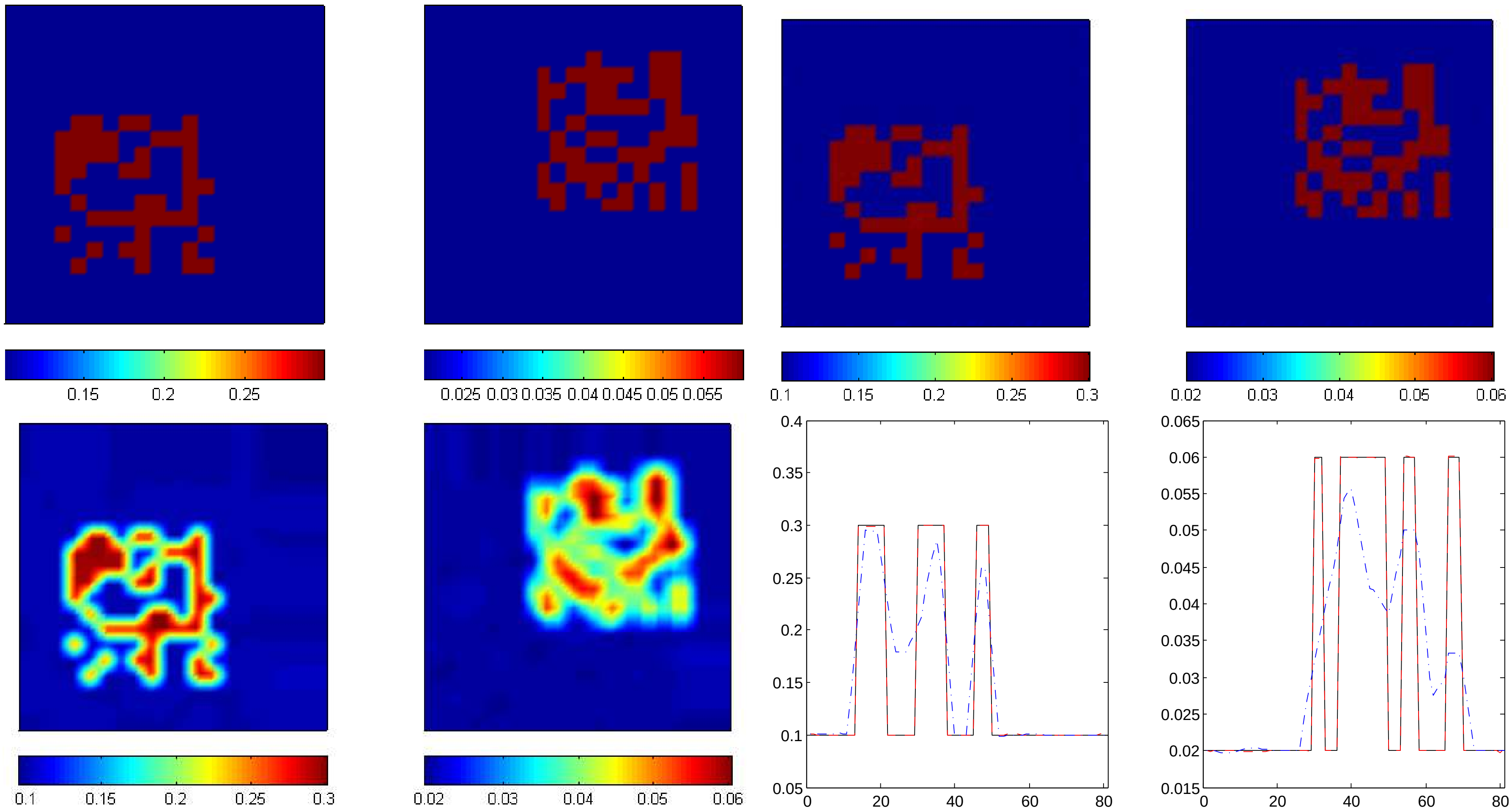
Plot of Internal functionals $H_{j=1,2}(x) = \sigma(x) u_{j=1,2}(x)$.

Explicit reconstructions $-\nabla \cdot \gamma \nabla u_j + \sigma u_j = 0$.

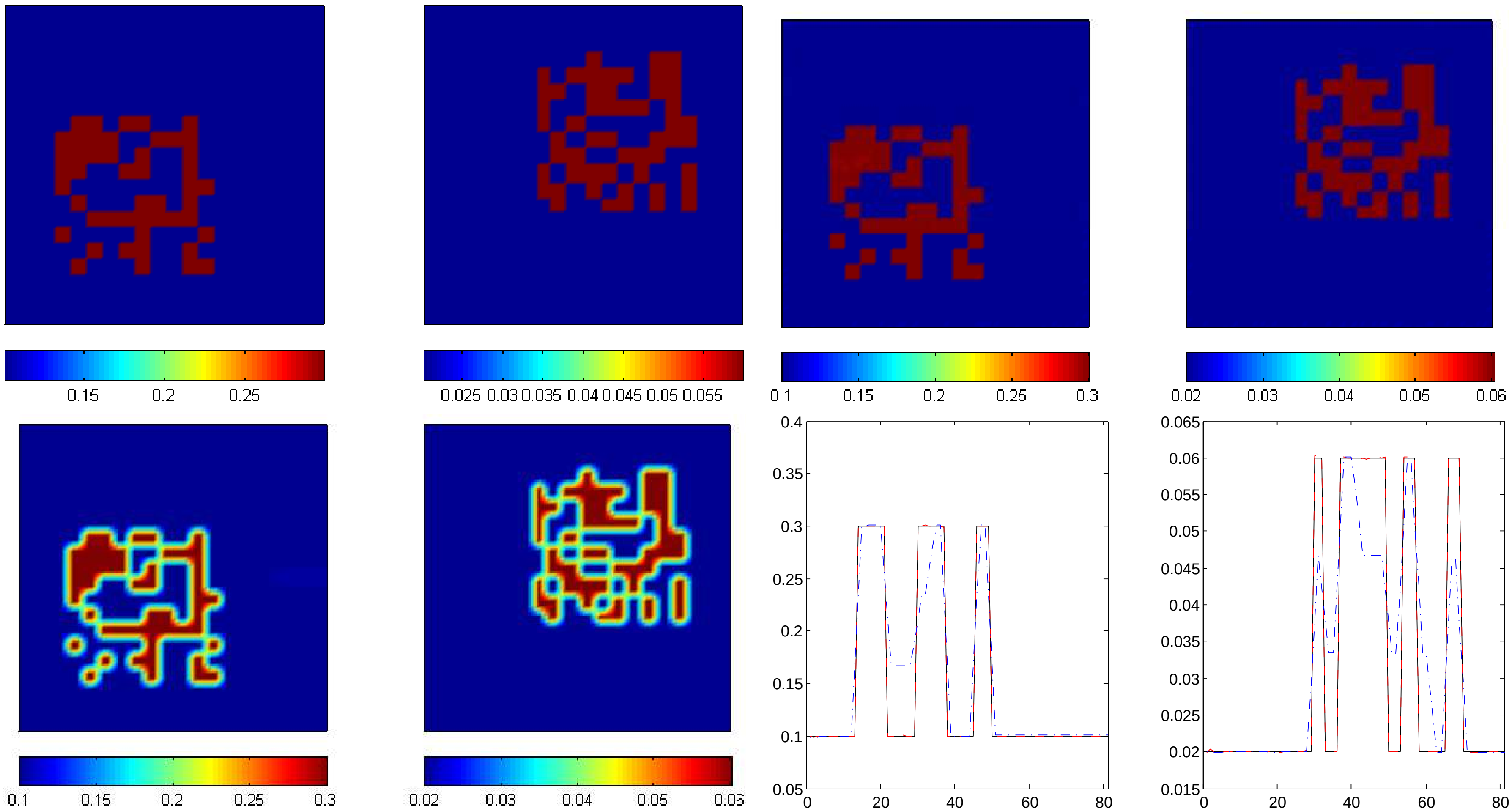


Explicit Reconstruction of (γ, σ) from functionals $H_{j=1,2} = \sigma u_{j=1,2}$.

QPAT reconstructions from two illuminations



QPAT reconstructions from multiple illuminations



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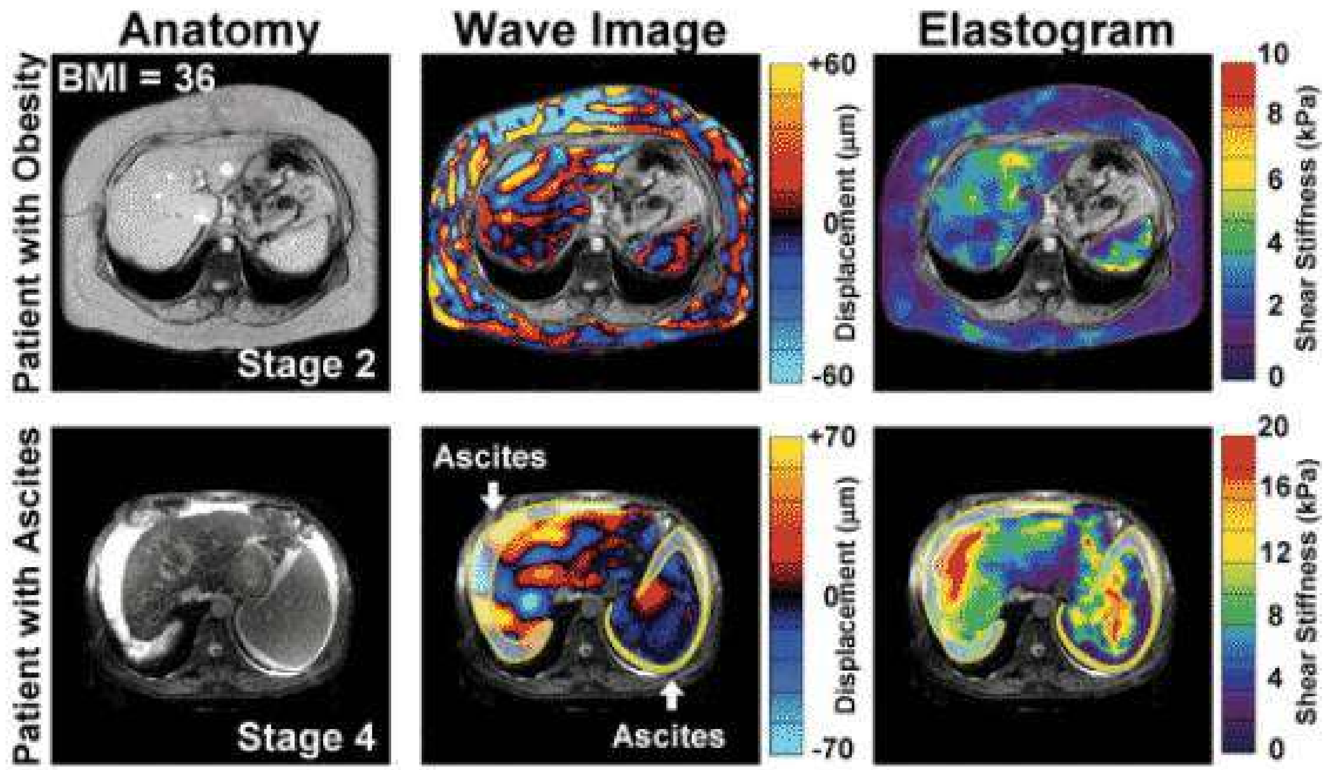
Elastography

High Contrast: Elastic properties

High Resolution Method 1: M.R.I. (Magnetic Resonance Elastography)

High Resolution Method 2: Ultrasound (Ultrasound Elastography)

Elastography and Magnetic Resonance



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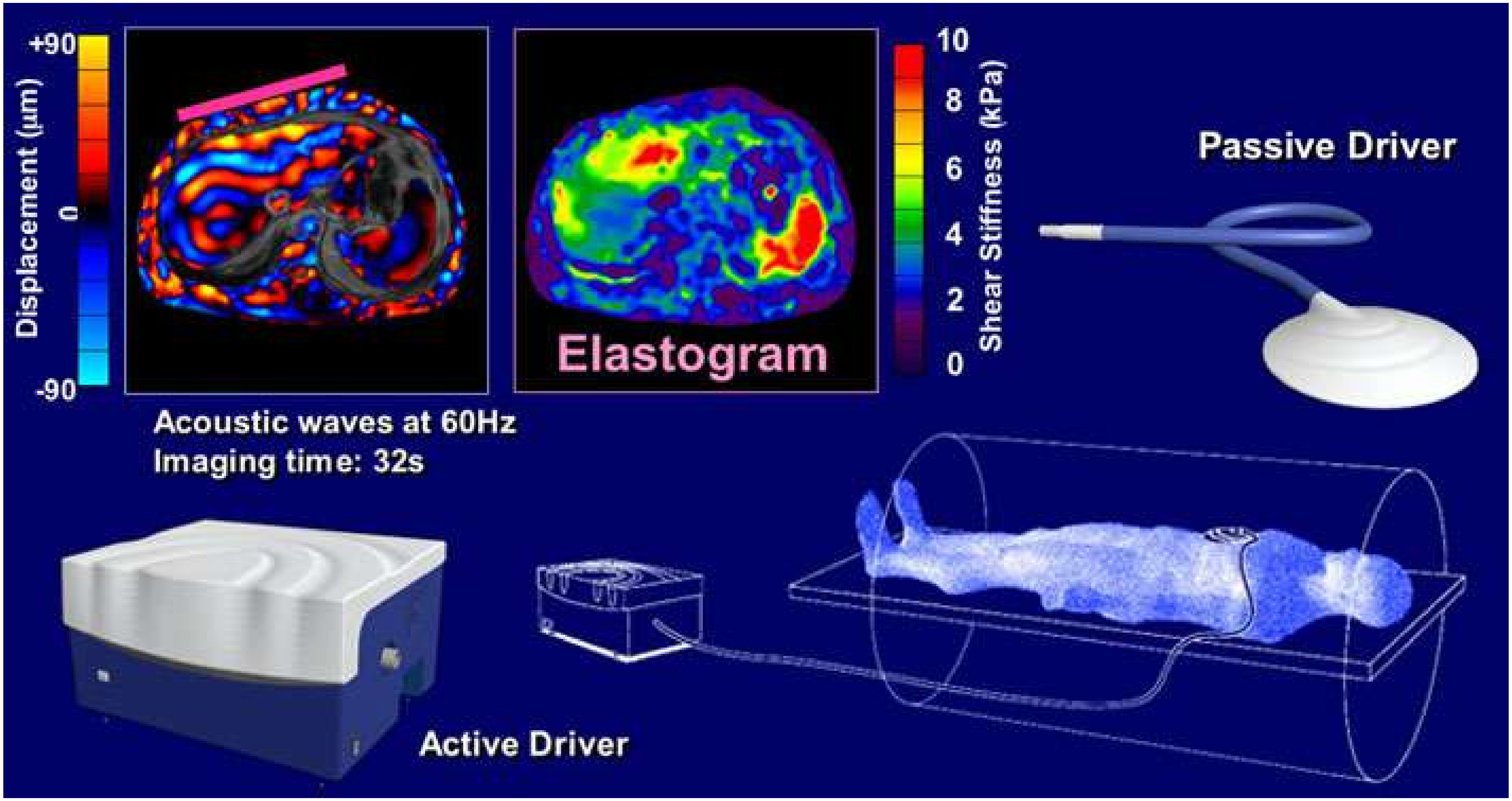
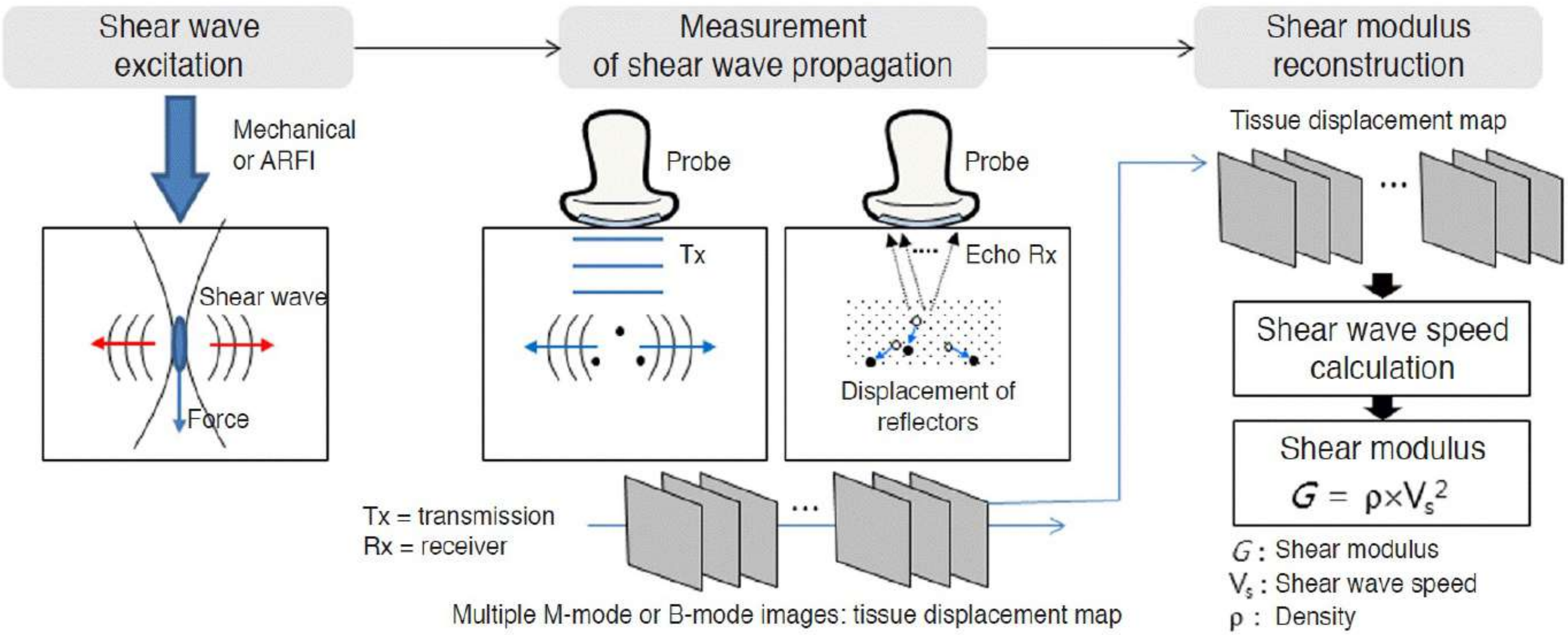
Ultrasound Elastography



Wave Generation, Probing & Reconstruction



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Physical processes

Propagating waves in body may be separated into two components.

(i) Slowly Propagating Shear Waves (m/s)

Referred to as Elastic Waves

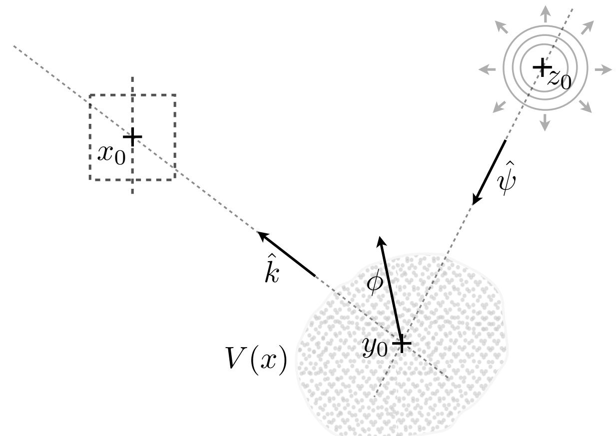
(ii) Rapidly Propagating Compressional Waves (km/s)

Referred to as Sound Waves (ultrasound)

The Slowly Propagating Elastic Waves generate **displacements** that are imaged by the probing Rapidly Propagating Sound Waves.

Joint works with Sébastien Imperiale and Pierre-David Létourneau.

Triangulation and geometry of acquisition



Sound propagation in heterogeneous medium in *single scattering approximation*:

$$u(t,x)=\int_0^t\int_{\mathbb{R}^6}G(s,x-y)V(y)G(t-s,y-z)(\Delta f)(z)dsdy;\ \ G(t,x)=\frac{\delta(t-|x|)}{4\pi|x|}.$$

Displacements of random scatterers $V(x)$ by $\tau(x)$: $V\to \underline{V(x+\tau(x))}$.

Phase-space localized measurements:

$$v(t_0,x_0,k)=\int_{\mathbb{R}^n}e^{-\frac{\alpha}{2}|x-x_0|^2}e^{-ik\cdot(x-x_0)}u(t_0,x)dx.$$

Asymptotic (high frequency) results

Assume a probing wavelength $\lambda \ll L$ the size of the domain. Then

$$v \sim \hat{V}_{y_0}(|k|\phi)\widehat{(\Delta f)}(|k|\hat{\psi})$$

and second measurement *after spatial shift* to

$$v_\tau \sim e^{i|k|\tau(y_0)\cdot\phi}\hat{V}_{y_0}(|k|\phi)\widehat{(\Delta f)}(|k|\hat{\psi}).$$

As a consequence, we have the explicit reconstruction procedure

$$\boxed{\frac{v_\tau}{v} \sim e^{i|k|\tau(y_0)\cdot\phi}}$$

provides an aliased (up to $2\pi/|k|$) estimate for $\underline{\tau(y_0)\cdot\phi}$ *locally* at y_0.

Spatial Resolution

The ratio of measurements provides an aliased version of $\tau(x)\cdot\phi$. Changing the source/detector geometry allows one to reconstruct *vector-valued* displacements $\tau(x)$.

The resolution of the method is at best of order $\sqrt{\varepsilon}$ with $\varepsilon=\frac{\lambda}{L}$. Precise calculations show that the available measurements are of the form

$$v_{\varepsilon\tau}\approx C_\varepsilon\int e^{i|k|\phi\cdot y}e^{-\frac{\alpha\varepsilon}{2}(\phi\cdot y)^2}e^{-\varepsilon\frac{|k|^2}{2\alpha}\Big(\Big|\frac{(I-\hat{k}\otimes\hat{k})y}{|y_0-x_0|}\Big|^2\Big)}V_{y_0}\Big(y+\tau(y_0+\varepsilon y)\Big)dy.$$

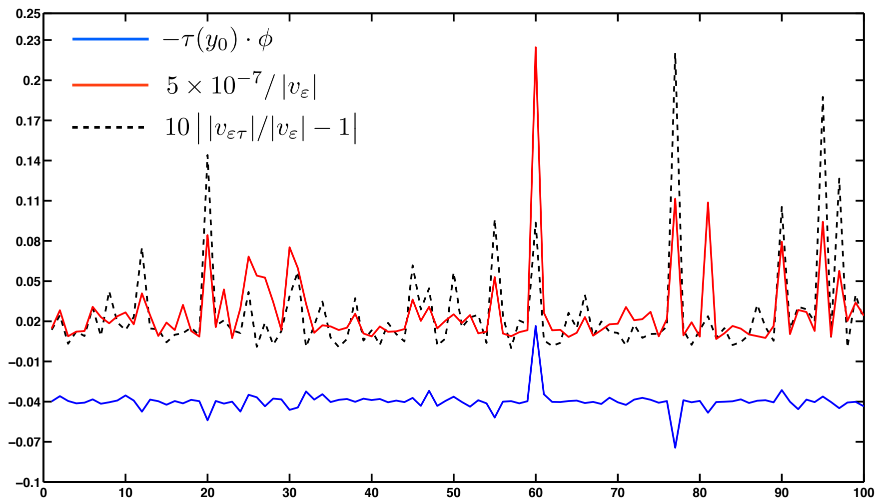
The support of this integral is roughly $\varepsilon^{-\frac{1}{2}}$ and so we need $|\sqrt{\varepsilon}\nabla\tau|\ll 1$ in order for the factor $e^{i|k|\tau(y_0)\cdot\phi}$ to appear.

Numerical simulations

Consider a vectorial displacement and $y_0 = (0, -2, 0)$.

$$\tau(y) = \frac{\varepsilon}{100}\Big(\cos(\pi\, y_1),\, 2\cos(\pi\, y_1),\, 0\Big), \qquad \tau(y_0)\cdot\phi = 0.04.$$

Reconstructions for several realizations of random medium are



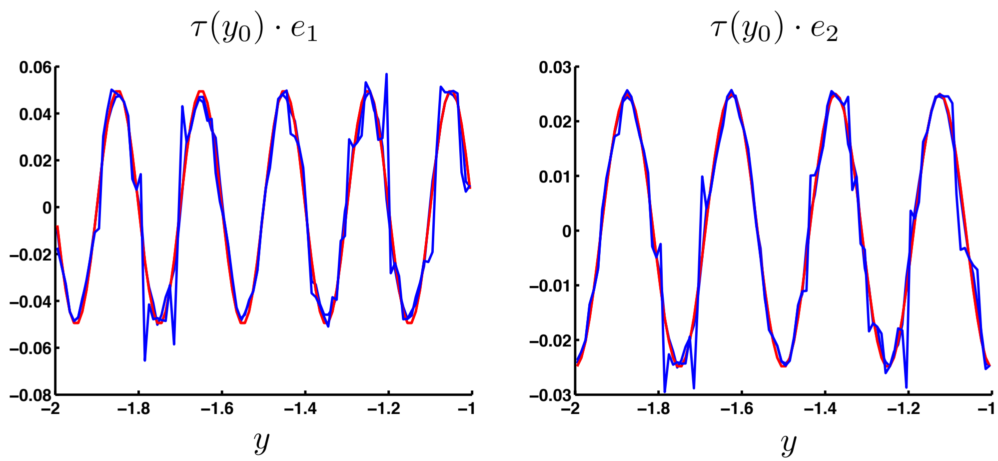
We observe good reconstructions except when v_ε is too small.

Numerical simulations

Consider the vectorial displacement

$$\tau(y) = \frac{\varepsilon}{100}\Big(\cos(\pi\, y_1),\, 2\,\cos(\pi\, y_1),\, 0\Big).$$

Reconstruction from $\frac{v_{\varepsilon\tau}}{v_\varepsilon} \sim e^{i|k|\tau(y_0)\cdot\phi}$ along a line segment

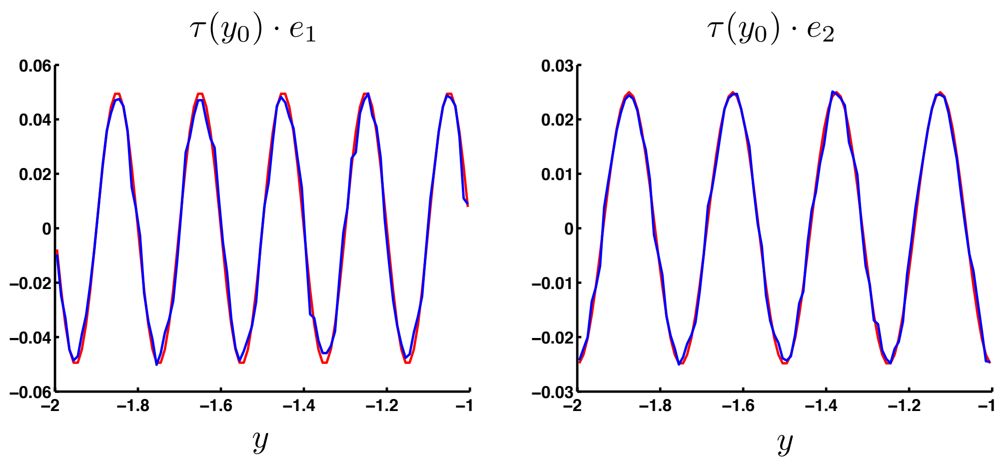


Numerical simulations

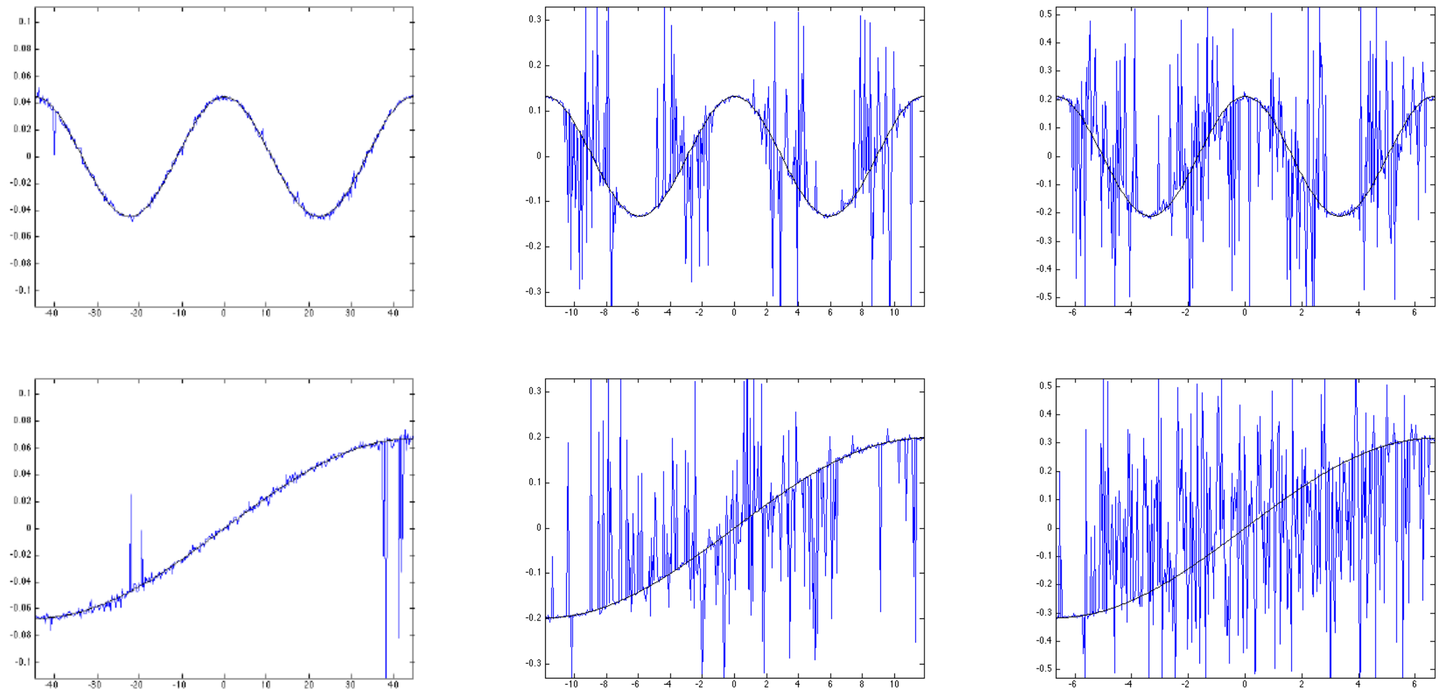
Consider the same vectorial displacement

$$\tau(y) = \frac{\varepsilon}{100}\Big(\cos(\pi\, y_1),\, 2\cos(\pi\, y_1),\, 0\Big).$$

Reconstruction from $\frac{v_{\varepsilon\tau}}{v_\varepsilon} \sim e^{i|k|\tau(y_0)\cdot\phi}$ selecting $|v_\varepsilon|$ "large" .



Limited resolution



Reconstruction (blue) and true value (black) of the x-displacement (top) and y-displacement (bottom) for $\phi_1(x)$ for *decreasing* $\epsilon = 1e^{-2}, 5e^{-2}, 1e^{-1}$ (left to right).

The reconstructions fail where the local variations are large.

Summary of displacement reconstruction

When $\sqrt{\varepsilon}|\nabla\tau| \ll 1$, then $\phi(t_0, x_0, z_0, k)\cdot\tau(x)$ can be reconstructed locally using the ratio of FBI transforms of sound wave measurements.

Aliasing occurs when τ/ε is large. Can be fixed with more measurements.

The denominator is away from 0 with high probability for reasonable distributions on V.

Triangulation in phase space (position and direction) limits the resolution to $\sqrt{\varepsilon}$ (uncertainty principle), where ε is the typical wavelength of the probing sound waves. (Can be overcome with *a lot* of measurements.)

Example of a functional of measurements $v_{\varepsilon\tau}/v_\varepsilon$ that is *statistically stable* and independent of the unrecoverable highly oscillatory potential V_ε.

Elastograms

Elastic displacements are imaged by sonic waves or magnetic resonance.

The second, quantitative, inverse problem aims to reconstruct the elastic properties of bodies from such displacements.

In elastography, displacements are solutions to systems of (linear or non-linear) equations of elasticity.

We first consider scalar second-order equations, joint work with G. Uhlmann CPAM 2013; and anisotropic systems of elasticity, joint work with F. Monard and G. Uhlmann 2015.

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Reconstructions from solution measurements

Consider a *general scalar elliptic* equation

$$\nabla \cdot a \nabla u + b \cdot \nabla u + c u = 0 \quad \text{in } X, \qquad u = f \quad \text{on } \partial X$$

with a, b, c, $\nabla \cdot a$ of class $C^{0,\alpha}(\bar{X})$ for $\alpha > 0$, complex-valued, and $\alpha_0 |\xi|^2 \leq \xi \cdot (\Re a)\xi \leq \alpha_0^{-1} |\xi|^2$. For τ a non-vanishing function on X, define

$$a_\tau = \tau a, \quad b_\tau = \tau b - a \nabla \tau, \quad c_\tau = \tau c$$

and the equivalence class $\mathfrak{c} := (a, b, c) \sim (a_\tau, b_\tau, c_\tau)$.

Let $I \in \mathbb{N}^*$ and $(f_i)_{1 \leq i \leq I}$ be I boundary conditions. Define $\mathfrak{f} = (f_1, \ldots, f_I)$. The **measurement operator** $\mathfrak{M}_{\mathfrak{f}}$ is

$$\mathfrak{M}_{\mathfrak{f}} : \quad \mathfrak{c} \mapsto \mathfrak{M}_{\mathfrak{f}}(\mathfrak{c}) = (u_1, \ldots, u_I),$$

with $H_j(x) = u_j(x)$ **solution** of the above elliptic problem with $f = f_j$.

Unique reconstruction up to gauge transformation

$$\nabla\cdot a\nabla u_j + b\cdot\nabla u_j + c u_j = 0 \quad \text{in } X, \qquad u_j = f_j \quad \text{on } \partial X, \quad 1\leq j\leq I.$$

We assume the above elliptic equation well posed for $\mathfrak{c}=(a,b,c)$.

Theorem [B. Uhlmann CPAM 2013]. Let $\mathfrak{c}$ and $\tilde{\mathfrak{c}}$ be two classes of coefficients with (a,b,c) and $\nabla\cdot a$ of class $C^{m,\alpha}(\bar X)$ for $\alpha>0$ and $m=0$ or $m=1$.

For I *sufficiently large* and an *open set of boundary conditions* $\mathfrak{f}=(f_j)_{1\leq j\leq I}$, then $\mathfrak{M}_{\mathfrak{f}}(\mathfrak{c})$ uniquely and stably determines $\mathfrak{c}$:

$$\begin{aligned}\|(a,b+\nabla\cdot a,c)-(\tilde a,\tilde b+\nabla\cdot\tilde a,\tilde c)\|_{W^{m,\infty}(X)} &\leq C\|\mathfrak{M}_{\mathfrak{f}}(\mathfrak{c})-\mathfrak{M}_{\mathfrak{f}}(\tilde{\mathfrak{c}})\|_{W^{m+2,\infty}(X)},\\ \|b-\tilde b\|_{L^\infty(X)} &\leq C\|\mathfrak{M}_{\mathfrak{f}}(\mathfrak{c})-\mathfrak{M}_{\mathfrak{f}}(\tilde{\mathfrak{c}})\|_{W^{3,\infty}(X)},\end{aligned}$$

for $m=0,1$ and for an appropriate $(\tilde a,\tilde b,\tilde c)$ of $\tilde{\mathfrak{c}}$.

Number of internal functionals

$$\nabla \cdot a \nabla u_j + b \cdot \nabla u_j + c u_j = 0 \quad \text{in } X, \qquad u_j = f_j \quad \text{on } \partial X, \quad 1 \le j \le I.$$

Results hold provided that # *of internal functionals* I is *sufficiently large*.

When global solutions can be constructed (for instance Complex Geometric Optics solutions), then we can show that

$$I = I_n = \frac{1}{2} n(n+3) \quad \text{when } a \text{ is a tensor}$$

$$I = I_n = n+1 \qquad \text{when } a \text{ is a scalar.}$$

In both cases, $\dim(a, b, c) = I_n + 1$ so I_n is optimal # of functionals.

In the general case with a a complex-valued tensor, only local solutions may be constructed. They are controlled from ∂X by a *Runge approximation* based on a Unique Continuation principle.

Boundary controls

The preceding stability estimates hold for *an open set* of boundary conditions $\mathfrak{f} = (f_1, \ldots, f_I)$. What one really requires is that the solution $\{u_i\}$ satisfy locally linear independence constraints. More precisely, we want that in the vicinity of a point x_0, the gradients $\{\nabla u_i\}$ and the Hessians $\{\nabla \otimes \nabla u_i\}$ form a family of *maximal rank*.

This is done as follows. We construct approximate local solution $\tilde{u}_j$ in the vicinity of x_0 on $B(x_0, r)$ for r small (think of perturbations of harmonic polynomials) that satisfy the maximal rank condition.

We then use the Runge approximation (a consequence of the unique continuation property for our elliptic equation) to obtain the (non-constructive) *existence of boundary conditions* $\mathfrak{f}$ such that the solutions u_j (and enough of their derivatives) are sufficiently close to $\tilde{u}_j$ and hence also satisfy the maximal rank condition. This imposes *smoothness constraints* on (a, b, c).

Unique reconstruction of the gauge

In some situations (as in Elastography), the gauge τ in $\mathfrak{c}$ can be *uniquely and stably* determined:

Corollary [B. Uhlmann CPAM 2013] When $b = 0$, then $\mathfrak{M}_{\mathfrak{f}}(\mathfrak{c})$ uniquely determines $(\gamma, 0, c)$. Define $\gamma = \tau M^0$ with $\mathrm{Det}(M^0) = 1$. Then we have the following stability result:

$$\|(\gamma, c) - (\tilde{\gamma}, \tilde{c})\|_{L^\infty(X)} \leq C\|\mathfrak{M}_{\mathfrak{f}}(\mathfrak{c}) - \mathfrak{M}_{\mathfrak{f}}(\tilde{\mathfrak{c}})\|_{W^{2,\infty}(X)}.$$

When M^0 is *known*, then we have the *more stable* reconstruction:

$$\|\tau - \tilde{\tau}\|_{W^{1,\infty}(X)} \leq C\|\mathfrak{M}_{\mathfrak{f}}(\mathfrak{c}) - \mathfrak{M}_{\mathfrak{f}}(\tilde{\mathfrak{c}})\|_{W^{2,\infty}(X)}.$$

The reconstruction of the determinant of γ is *more stable* than the reconstruction of the anisotropy of the possibly complex valued tensor γ. This has been observed numerically in different settings.

Generalization to TE / PAT settings with $b = 0$

$$\nabla \cdot a \nabla u_j + c u_j = 0 \quad \text{in} \quad X, \qquad u_j = f \quad \text{on} \quad \partial X, \qquad 1 \le j \le J.$$

$$H_j^{UE} = u_j, \qquad H_j^{PAT} = \Gamma\, c\, u_j, \qquad H_j^{TAT} = \Gamma\, \Im c\, u_j u_1^*.$$

Decompose $a = B^2 \hat{a}$ with $\det \hat{a} = 1$. Assume J sufficiently large. Then:

$$(H_j^{UE})_{1 \le j \le J} \quad \Longrightarrow \quad (a, c) \quad \Longrightarrow \quad \text{any } H^{UE}$$

$$(H_j^{PAT})_{1 \le j \le J} \quad \Longrightarrow \quad \Big(\hat{a}, \frac{\Gamma c}{B}, \frac{\nabla \cdot \hat{a} \nabla B}{B} + \frac{c}{B^2}\Big) \quad \Longrightarrow \quad \text{any } H^{PAT}$$

$$(H_j^{TAT})_{1 \le j \le J} \quad \Longrightarrow \quad \Big(\hat{a}, \Gamma \frac{\Im c}{|B|^2}, \frac{\nabla \cdot \hat{a} \nabla B}{B} + \frac{c}{B^2}\Big) \quad \Longrightarrow \quad \text{any } H^{TAT}$$

QPAT: When Γ known a priori, then (a, c) stably reconstructed.

QTAT: When a real-valued, Γ always (stably) reconstructed, but not $(B, \Re c, \Im c)$. When $a = I$, then $(\Gamma, \Re c, \Im c)$ stably reconstructed.

Anisotropic Elasticity

Consider the reconstruction of anisotropic tensor $C = \{C_{ijkl}\}_{1\leq i,j,k,l\leq 3}$ ($C_{ijkl} = C_{jikl} = C_{ijlk} = C_{klij}$) from knowledge of a finite number of displacement fields $\{\mathbf{u}^{(j)}\}_{j\in J}$, solutions of the linear elasticity equation

$$\nabla\cdot(C:(\nabla\mathbf{u}+(\nabla\mathbf{u})^T))=0 \qquad (X), \qquad \mathbf{u}|_{\partial X}=\mathbf{g} \quad \text{(prescribed)}.$$

There are 21 unknown components.

Define $\epsilon = \frac{1}{2}(\nabla\mathbf{u}+(\nabla\mathbf{u})^T)$. When a sufficiently large number of $\epsilon^{(j)}$ are known, then C can be uniquely and stably reconstructed.

Assumptions of independence

Assume the existence of 6 solutions such that for $\Omega \subset X$

$$\inf_{x\in\Omega} \det_V(\varepsilon^{(1)}(x), \dots, \varepsilon^{(6)}(x)) \geq c_0 > 0, \quad \text{for some constant } c_0.$$

Assume also that there exists N additional solutions $\mathbf{u}^{6+1}, \dots, \mathbf{u}^{6+N}$ giving rise to a family M of $3N$ matrices whose expressions are explicit in terms of $\{\varepsilon^{(j)}, \partial_\alpha \varepsilon^{(j)},\ 1 \leq \alpha \leq 3,\ 1 \leq j \leq 6+N\}$ such that

$$\inf_{x\in\Omega} \sum_{M'\subset M,\ \#M'=20} \mathbb{N}(M') : \mathbb{N}(M') \geq c_1 > 0, \quad \text{for some constant } c_1,$$

for $\mathbb{N}$ generalizing cross product $\mathbb{N}(M) := \frac{1}{\det(\mathbf{m}_1,\cdots,\mathbf{m}_{21})} \begin{vmatrix} M_1 : \mathbf{m}_1 & \cdots & M_1 : \mathbf{m}_{21} \\ \vdots & \ddots & \vdots \\ M_{20} : \mathbf{m}_1 & \cdots & M_{20} : \mathbf{m}_{21} \\ \mathbf{m}_1 & \cdots & \mathbf{m}_{21} \end{vmatrix}$ for $\mathbf{m}_{1\leq j\leq 21}$ a basis of $S_6(\mathbb{R})$.

Reconstruction results

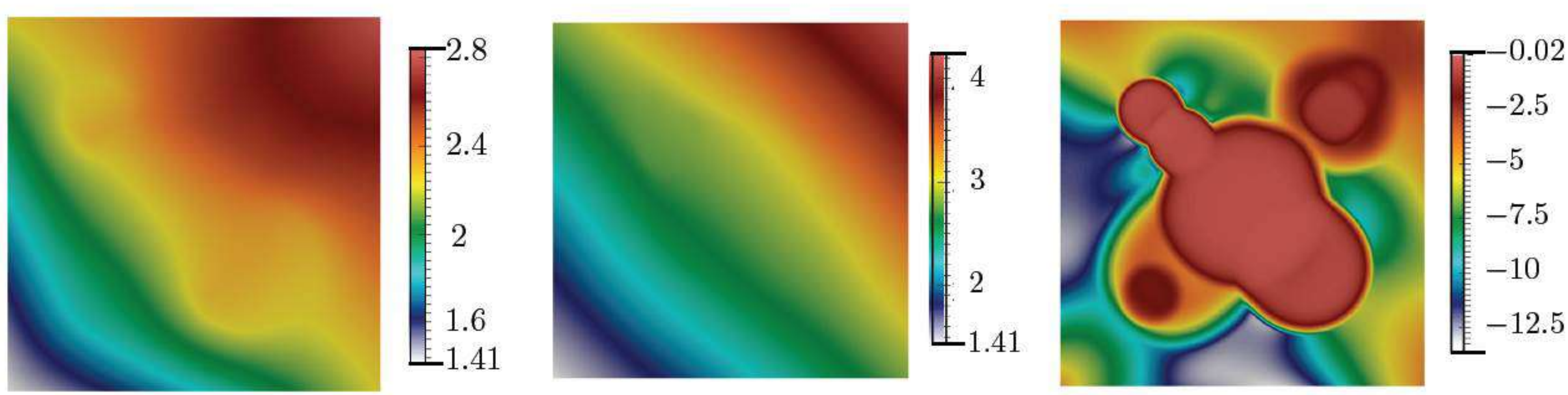
Theorem [B. Monard Uhlmann-2015] Assuming the above assumptions hold for $\{\mathbf{u}^{(j)}\}_{j=1}^{6+N}$ and $\{\mathbf{u}'^{(j)}\}_{j=1}^{6+N}$ corresponding to elasticity tensors C and C'. Then C and C' can each be uniquely reconstructed over Ω from knowledge of their corresponding solutions, with the following stability estimate for every integer $p \geq 0$

$$\|C - C'\|_{W^{p,\infty}(\Omega)} + \|\text{div}C - \text{div}C'\|_{W^{p,\infty}(\Omega)} \leq K \sum_{j=1}^{N+6} \|\epsilon^{(j)} - \epsilon'^{(j)}\|_{W^{p+1,\infty}(\Omega)}$$

If $C = \tau\tilde{C}$ for $\tilde{C}$ known, then

$$\|\tau - \tau'\|_{W^{p+1,\infty}(\Omega)} \leq K \sum_{j=1}^{N+6} \|\varepsilon^{(j)} - \varepsilon'^{(j)}\|_{W^{p+1,\infty}(\Omega)}.$$

2d Reconstructions in isotropic elasticity

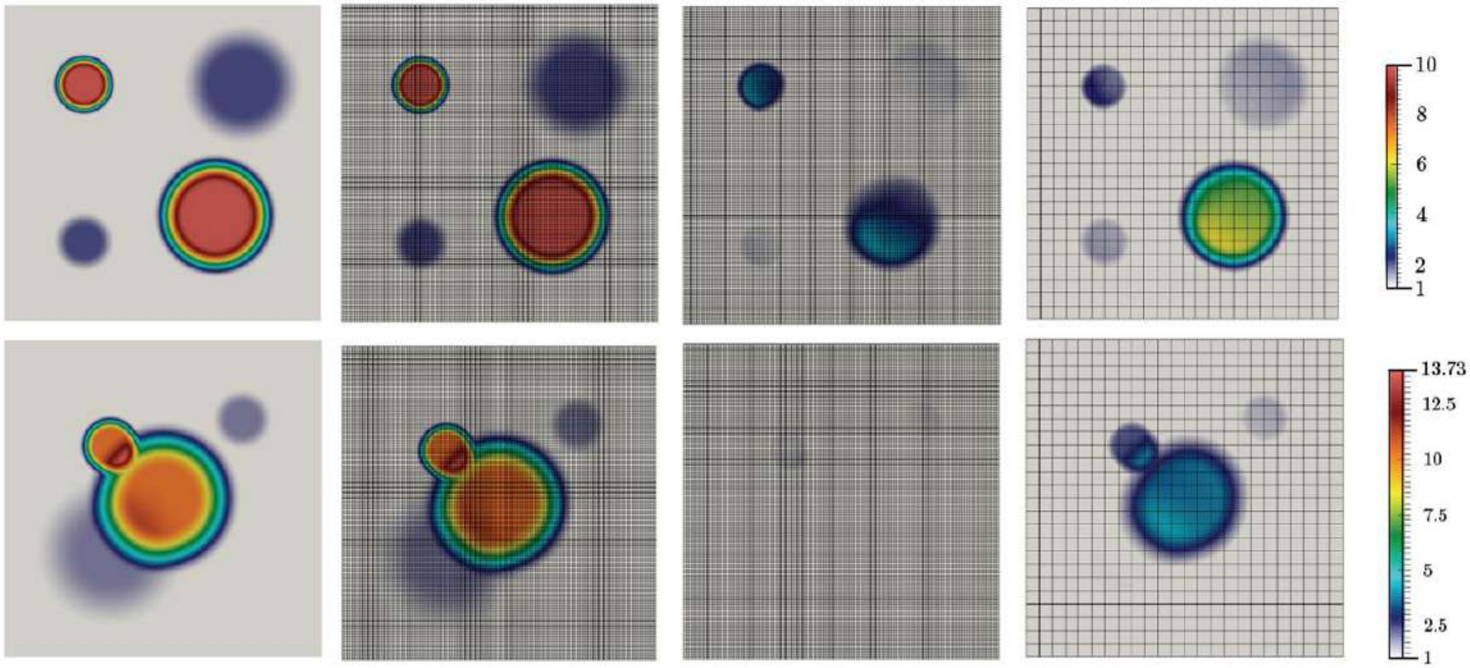


(a) $|\boldsymbol{u}_1(\boldsymbol{x})|$ (b) $|\boldsymbol{u}_2(\boldsymbol{x})|$ (c) $\det \mathscr{E}(\boldsymbol{x})$

Amplitude and determinant of two elastic displacements $\mathbf{u}_1$ and $\mathbf{u}_2$.

This and next pictures from B. Bellis Imperiale Monard IP 2014.

2d Reconstructions in isotropic elasticity

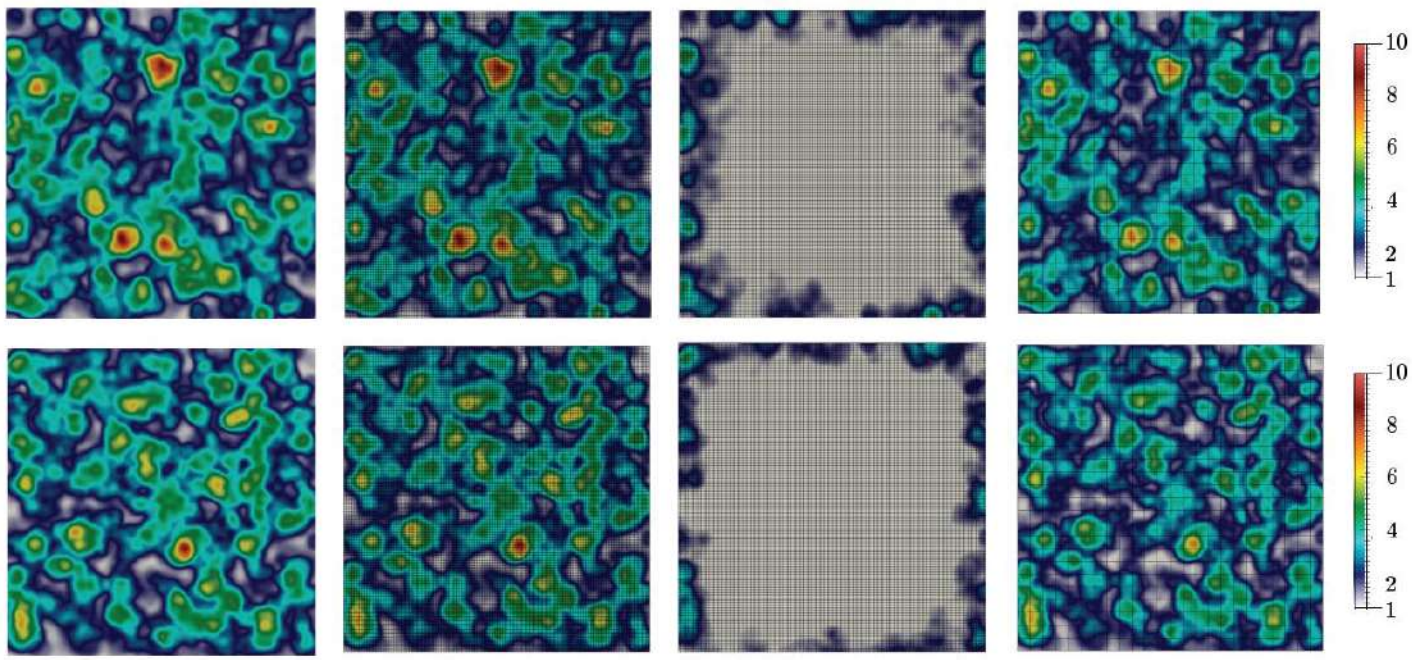


(a) $(\alpha(\boldsymbol{x}), \beta(\boldsymbol{x}))$ (b) $\delta = 0, k = 1$ (c) $\delta = 10^{-7}, k = 1$ (d) $\delta = 10^{-7}, k = 5$

Figure 6: (a) Exact values of $\alpha(\boldsymbol{x})$ *(top)* and $\beta(\boldsymbol{x})$ *(bottom)*; Corresponding reconstructions with (b) no noise nor regularization, (c) with noise but no regularization, (d) with noise and regularization.

Reconstruction of two Lamé parameters from displacements $\mathbf{u}_1$ and $\mathbf{u}_2$.

2d Reconstructions in isotropic elasticity



(a) $(\alpha(\boldsymbol{x}), \beta(\boldsymbol{x}))$ (b) $\delta = 0, k = 1$ (c) $\delta = 10^{-6}, k = 1$ (d) $\delta = 10^{-6}, k = 5$

Figure 7: (a) Exact values of $\alpha(\boldsymbol{x})$ *(top)* and $\beta(\boldsymbol{x})$ *(bottom)*; Corresponding reconstructions with (b) no noise nor regularization, (c) with noise but no regularization, (d) with noise and regularization.

Reconstruction of more heterogeneous Lamé parameters.

Other Hybrid Inverse Problems and Elliptic Theory

High Contrast: Electrical, Elastic, or Optical

High Resolution: MRI or Ultrasound.

Examples of Hybrid Inverse Problems

- Examples of PDE models for High-contrast coefficients:

$$-\nabla\cdot\gamma(x)\nabla u+\sigma(x)u=0 \text{ in } X, \qquad u=f \text{ on } \partial X$$
$$-\nabla\times\nabla\times E+n(x)k^2E+i\sigma(x)E=0 \text{ in } X, \qquad \nu\times E=f \text{ on } \partial X$$
$$-\nabla\cdot C:(\nabla\mathbf{u}+(\nabla\mathbf{u})^T)=0 \text{ in } X, \qquad \mathbf{u}=\mathbf{g} \text{ on } \partial X$$

- In Step 1, High-Resolution modality provides **Internal functionals** :

$H(x)$	$=$	$\Gamma(x)\sigma(x)\,u(x)$	*Photo-acoustics*
$H(x)$	$=$	$u(x)$ or $\mathbf{u}(x)$	Elastography
$H(x)$	$=$	$\sigma(x)\,\lvert u\rvert^2(x)$ or $\sigma(x)\lvert E\rvert^2(x)$	Thermo-acoustics
$H(x)$	$=$	$\boxed{\gamma(x)\,\nabla u(x)\cdot\nabla u(x)}$	*Ultrasound Modulation*
$H(x)$	$=$	$\gamma(x)\,\nabla u(x)$ or $\gamma(x)\,\lvert\nabla u(x)\rvert$	CDII, MREIT

- One or several **illuminations** $f=f_j$ (and thus $H=H_j$) for $1\leq j\leq J$.

Theoretical analyses of HIP

Can we find *general theories* for stability/uniqueness of (many) HIPs?
Can we understand role of number of measurements J, of B.C. f_j?

Consider as an example the UMT problem

$$
\begin{aligned}
-\nabla\cdot\gamma(x)\nabla u_1 &= 0 \quad \text{in } X, \qquad u_1 = f_1 \text{ on } \partial X\\
-\nabla\cdot\gamma(x)\nabla u_2 &= 0 \quad \text{in } X, \qquad u_2 = f_2 \text{ on } \partial X\\
H_1(x) &= \gamma(x)\nabla u_1(x)\cdot\nabla u_1(x) \quad \text{in } X\\
H_2(x) &= \gamma(x)\nabla u_2(x)\cdot\nabla u_2(x) \quad \text{in } X
\end{aligned}
$$

Theoretical analyses of HIP

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\gamma(x) \nabla u_1(x) \cdot \nabla u_1(x) &= H_1(x) \quad \text{in } X \\
\gamma(x) \nabla u_2(x) \cdot \nabla u_2(x) &= H_2(x) \quad \text{in } X
\end{aligned}$$

Theoretical analyses of HIP

Can we find *general theories* for stability/uniqueness of (many) HIPs? Can we understand role of number of measurements J, of B.C. f_j?

Consider an Ultrasound Modulation Tomography (UMT) problem

$$\begin{aligned}
-\nabla\cdot\gamma(x)\nabla u_1 &= 0 && \text{in } X, && u_1 = f_1 \text{ on } \partial X\\
-\nabla\cdot\gamma(x)\nabla u_2 &= 0 && \text{in } X, && u_2 = f_2 \text{ on } \partial X\\
\gamma(x)\nabla u_1(x)\cdot\nabla u_1(x) &= H_1(x) && \text{in } X\\
\gamma(x)\nabla u_2(x)\cdot\nabla u_2(x) &= H_2(x) && \text{in } X.
\end{aligned}$$

The left-hand side is a **polynomial** of γ, u_j and their derivatives. This forms a 4×3 **redundant system of nonlinear PDEs** in X.

Systems of coupled nonlinear equations

Hybrid inverse problems may be recast as the **system of PDE**:

$$\mathcal{F}(\gamma, \{u_j\}_{1\leq j\leq J}) = \mathcal{H}, \qquad (1)$$

where γ are unknown parameters and u_j are PDE solutions.

For UMEIT, we have

$$\mathcal{F}(\gamma, \{u_j\}_{1\leq j\leq J}) = \begin{pmatrix} -\nabla\cdot\gamma\nabla u_j \\ \gamma|\nabla u_j|^2 \end{pmatrix}, \qquad \mathcal{H} = \begin{pmatrix} 0 \\ H_j \end{pmatrix}, \qquad 2J - \text{ rows }.$$

(1) is a possibly **redundant** $2J\times(J+m)$ **system of nonlinear equations** with $J+m$ unknowns ($m=1$ if γ is scalar).

HIP theory concerns uniqueness, stability, reconstruction procedures for *typically redundant (over-determined) systems* of the form (1) with appropriate boundary conditions.

The $0-$Laplacian with $J=1$

$$-\nabla\cdot\gamma(x)\nabla u=0,\qquad \gamma(x)|\nabla u|^2(x)-H(x)=0\qquad u=f \text{ on } \partial X.$$

The elimination of γ yields the 0-*Laplacian*

$$-\nabla\cdot\frac{H(x)}{|\nabla u|^2}\nabla u=0 \text{ in } X,\qquad u=f \text{ on } \partial X.$$

The above equation *with Cauchy data* may be transformed as

$$(I-2\widehat{\nabla u}\otimes\widehat{\nabla u}):\nabla^2 u+\nabla\ln H\cdot\nabla u=0 \text{ in } X,\quad u=f \text{ and } \frac{\partial u}{\partial\nu}=j \text{ on } \partial X.$$

Here $\widehat{\nabla u}=\frac{\nabla u}{|\nabla u|}$. This is a quasilinear strictly hyperbolic equation with $\widehat{\nabla u}(x)$ a "time-like" direction. Cauchy data generate stable solutions on "space-like" part of ∂X for the *Lorentzian metric* $(I-2\widehat{\nabla u}\otimes\widehat{\nabla u})$.

Stability on domain of influence

Let u and $\tilde{u}$ be two solutions of the hyperbolic equation and $v = u - \tilde{u}$.

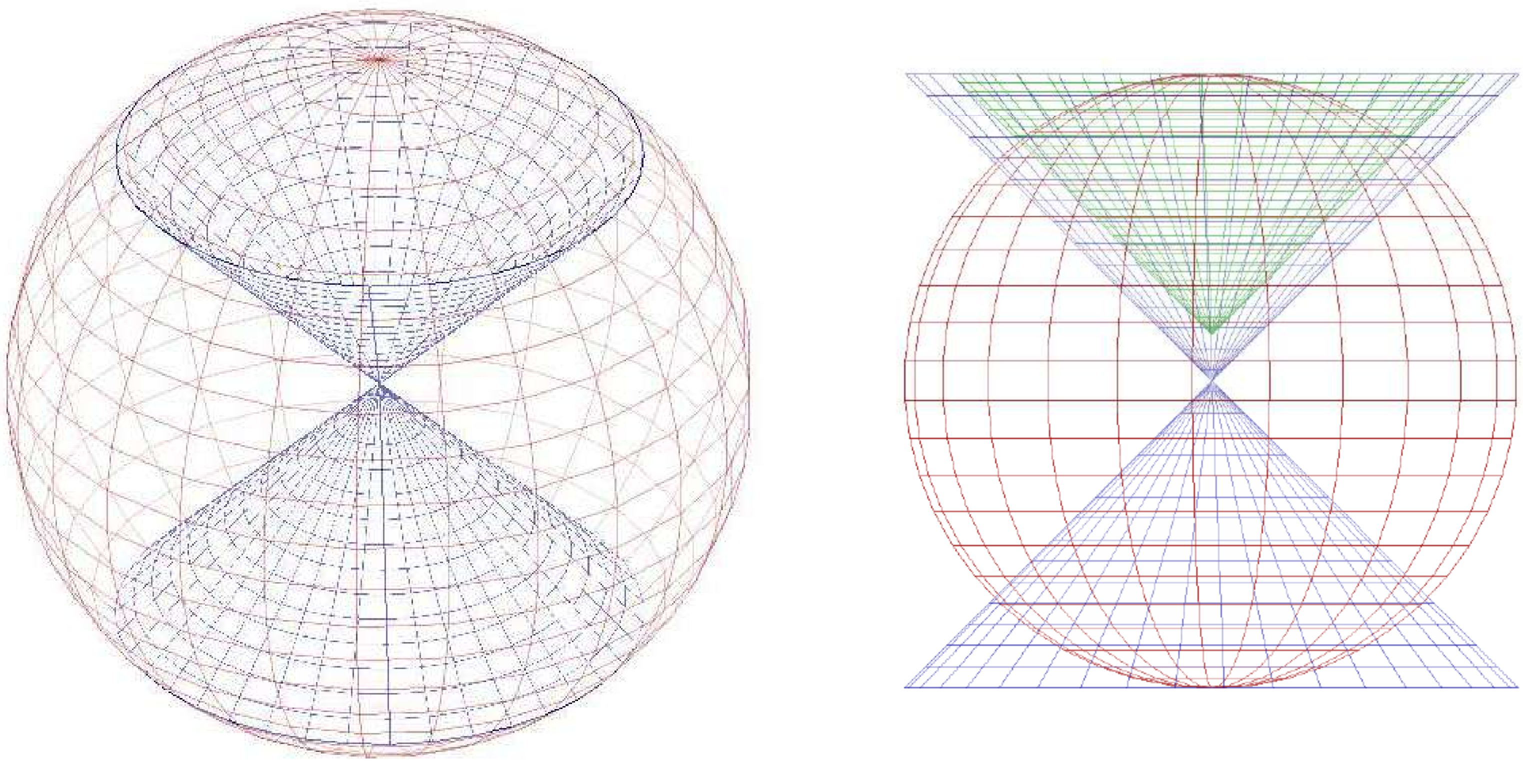
IF (appropriate) Lorentzian metric is uniformly strictly hyperbolic, then:

Theorem [B. Anal&PDE 13]. Let $\Sigma_1 \subset \Sigma_g$ space-like component of ∂X and $\mathcal{O}$ **domain of influence** of Σ_1. For θ distance of $\mathcal{O}$ to boundary of domain of influence of Σ_g, we have the local stability result:

$$\int_{\mathcal{O}} |v|^2 + |\nabla v|^2 + (\gamma - \tilde{\gamma})^2 \, dx \leq \frac{C}{\theta^2} \Big(\int_{\Sigma_1} |\delta f|^2 + |\delta j|^2 \, d\sigma + \int_{\mathcal{O}} |\nabla \delta H|^2 \, dx \Big),$$

where $\gamma = \frac{H}{|\nabla u|^2}$ and $\tilde{\gamma} = \frac{\tilde{H}}{|\nabla \tilde{u}|^2}$. We observe the **loss of one derivative** from δH to $\delta\gamma$ (sub-elliptic estimate).

Domain of Influence



Domain of influence (blue) for metric $g = I - 2e_z \otimes e_z$ on sphere (red). Null-like vectors (surface of cone) generate instabilities. Right: Sphere (red), domains of uniqueness (blue) and with controlled stability (green).

Elliptic Theory for multiple measurements

Consider the system

$$-\nabla\cdot\gamma(x)\nabla u_j=0,\quad \gamma(x)|\nabla u_j|^2(x)=H_j(x),\quad u_j|_{\partial X}=f_j,\quad 1\le j\le J.$$

- With $J=1$, the system is hyperbolic.
- With $J\ge 2$, the **redundant** system $2J\times(J+1)$ may be **elliptic**.
- After *linearization*, we obtain the system:

$$\begin{aligned}\nabla\cdot\delta\gamma\nabla u_j+\nabla\cdot\gamma\nabla\delta u_j &= 0 &(2)\\ \delta\gamma|\nabla u_j|^2+2\gamma\nabla u_j\cdot\nabla\delta u_j &= \delta H_j. &(3)\end{aligned}$$

With $v=(\delta\gamma,\delta u_1,\ldots,\delta u_J)$, we recast the above system for v as

$$\mathcal{A}v:=(\mathcal{P}_J+\mathcal{R}_J)v=\mathcal{S}$$

where $\mathcal{P}_J$ is the *principal part* and $\mathcal{R}_J$ is lower order.

Let us define $F_j = \nabla u_j$. The symbol of $\mathcal{P}_J$, a $2J \times (J+1)$ system is:

$$\mathfrak{p}_J(x,\xi) = \begin{pmatrix} |F_1|^2 & 2\gamma F_1 \cdot i\xi & \dots & 0 \\ F_1 \cdot i\xi & -\gamma|\xi|^2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ |F_J|^2 & 0 & \dots & 2\gamma F_J \cdot i\xi \\ F_J \cdot i\xi & 0 & \dots & -\gamma|\xi|^2 \end{pmatrix}.$$

• System said *elliptic* when $\mathfrak{p}_J(x,\xi)$ **maximal rank** $(J+1)$ for all $\xi \in \mathbb{S}^{n-1}$.

(i) Redundant concatenation of hyperbolic systems $(J = 1)$ may be *elliptic*.

(ii) $\mathfrak{p}_J$ elliptic **IF** we choose f_j s.t. the following qualitative statement on quadratic forms holds: $\Big\{|\xi|^2 - 2(\hat{F}_j \cdot \xi)^2 = 0,\ 1 \le j \le J\Big\}$ implies $\xi = 0$.

For *ellipticity*, we thus want the light cones generated by the directions $\hat{F}_j$ to intersect to $\{0\}$. (shown to hold for appropriate boundary conditions f_j for instance using the method of CGO solutions.)

Theory of Redundant elliptic systems

• The system is elliptic in the sense of Douglis and Nirenberg.

Each row and column is given an index s_i and t_j and the principal term is the homogeneous differential operator of order $s_i + t_j$. For the above system, we choose $s_{2k+1} = 0$, $s_{2k} = 1$, $t_1 = 0$, $t_{k\geq 2} = 1$.

• We need boundary conditions that satisfy the Lopatinskii condition. Dirichlet conditions on δu_j and no condition on $\delta\gamma$ satisfy the LC.

Indeed, we need to show that $v(z) = (\delta\gamma(z), \ldots, \delta u_J(z)) \equiv 0$ is the only solution to

$$\delta u_j(0) = 0, \quad F_j \cdot N\partial_z\delta\gamma + \gamma\partial_z^2\delta u_j = 0, \; |F_j|^2\delta\gamma + 2\gamma F_j \cdot N\partial_z\delta u_j = 0, \; z > 0$$

vanishing as $z \to \infty$ for $N = \nu(x)$ at $x \in \partial X$ and z coordinate along $-N$. We observe that this is the case if $|F_j|^2 - 2(F_j \cdot N)^2 \neq 0$ for some j. This is the condition for joint ellipticity.

• Theory of Agmon-Douglis-Nirenberg extended to over-determined systems by Solonnikov shows that $Av = S$ (including boundary conditions) admits a left-parametrix R so that $RA = I - T$ with T compact.

Elliptic stability estimates

From the ADN-Sol. theory results the **Stability estimates**

$$\sum_{j=1}^{J+1} \|v_j\|_{H^{l+t_j}(X)} \leq C \sum_{i=1}^{2J} \|S_i\|_{H^{l-s_i}(X)} + C_2 \sum_{t_j>0} \|v_j\|_{L^2(X)}.$$

For the UMEIT example ($H_j = \gamma|\nabla u_j|^2$), this is:

$$\|\delta\gamma\|_{H^l(X)} + \sum_j \|\delta u_j\|_{H^{l+1}(X)} \leq C \sum_j \|\delta H_j\|_{H^l(X)} + C_2 \sum_j \|\delta u_j\|_{L^2(X)}.$$

• *No loss of derivatives* from δH to $\delta\gamma$: Optimal Stability (unlike $J=1$).

• We *do not have injectivity* of the system ($C_2 \neq 0$): A can be inverted up to a finite dimensional kernel with RA Fredholm of index 0.

Injectivity: Holmgren, Carleman, and Calderón

• Assume $\mathcal{A}$ is *elliptic* in the regular sense, i.e., $t_j = t$ and $s_i = 0$. Consider, with $t = 2$, the two problems

$$\mathcal{A}v = S, \quad v|_{\partial X} \text{ known}, \quad \text{and} \quad \mathcal{A}^t\mathcal{A}v = \mathcal{A}^t S, \quad v|_{\partial X} \ \& \ \partial_\nu v|_{\partial X} \text{ known}.$$

The second system is $(J+1)\times(J+1)$- determined even if the first one is $2J \times (J+1)$ redundant. It provides an *explicit* reconstruction procedure. Moreover, (non-)injectivity of the second one implies (non-)injectivity of the redundant (both in X and on ∂X) system:

$$\mathcal{A}v = 0, \quad v|_{\partial X} = \partial_\nu v|_{\partial X} = 0.$$

• **Injectivity** for such a system can be proved by Holmgren's theorem when $\mathcal{A}$ has analytic coefficients and by Carleman estimates, as obtained for systems in Calderón's theorem, for a restricted class of operators $\mathcal{A}$. Details in: *B. Contemp. Math. 2014*.

Holmgren and local results

Holmgren's theorem used for $\mathcal{A}$ with analytic coefficients and constant coefficient PDE theory used for $\mathcal{A}$ on a sufficiently small domain X.

When $\mathcal{A} = \mathcal{A}_A$ has analytic coefficients and $\mathcal{A}_A v = 0$, then an application of Hörmander's theorem shows that $WF_A(v) \subset WF_A(\det(\mathcal{A}_A^t \mathcal{A}_A)v)$ so that v is analytic. With vanishing Cauchy data, $v = 0$ and *injectivity* follows.

This provides *genericity* for hybrid inverse problems (invertibility of linear and nonlinear IP on open, dense, set).

When the spatial domain X is small, write $\mathcal{A} = \mathcal{A}_0 + (\mathcal{A} - \mathcal{A}_0)$ with $\mathcal{A}_0$ the operator with coefficients frozen at $x = 0$. We then apply the elliptic theory for constant coefficient operators to $\mathcal{A}_0$ and then to $\mathcal{A}$ by perturbation on a small domain.

Carleman estimates and Calderón's theorem

• When $\mathcal{A}$ is not analytic and X is not small, proving **injectivity** is *significantly more difficult* and may rely on Unique Continuation Principles.

Recalling that $\mathcal{A} = \mathcal{P} + \mathcal{R}$ with $\mathcal{P}$ leading term, we seek *injectivity results* depending on leading term $\mathcal{P}$ and not $\mathcal{R}$. This essentially forces $\mathfrak{p}(\xi + \tau N)$ for $\xi \in \mathbb{S}^{n-1}$ and $N \in \mathbb{S}^{n-1}$ to be a diagonal (diagonalized) symbol with diagonal terms that are polynomials in τ with at most simple real roots and at most double complex roots.

• Applies to modified form of ultrasound modulation problem and systems of the form $\begin{pmatrix} P_1 & C \\ 0 & P_2 \end{pmatrix} u = 0$ with P_1 satisfying UCP, P_2 elliptic with *simple complex roots* (saving one to control C; all operators of order m here).

Details in: *B. Contemp. Math. 2014*.

Invertibility and Local Uniqueness for Nonlinear I.P.

Recast original nonlinear I.P. as

$$\mathcal{F}(v_0+v)=\mathcal{H}, \qquad \mathcal{H}_0 := \mathcal{F}(v_0), \qquad A=\mathcal{F}'(v_0).$$

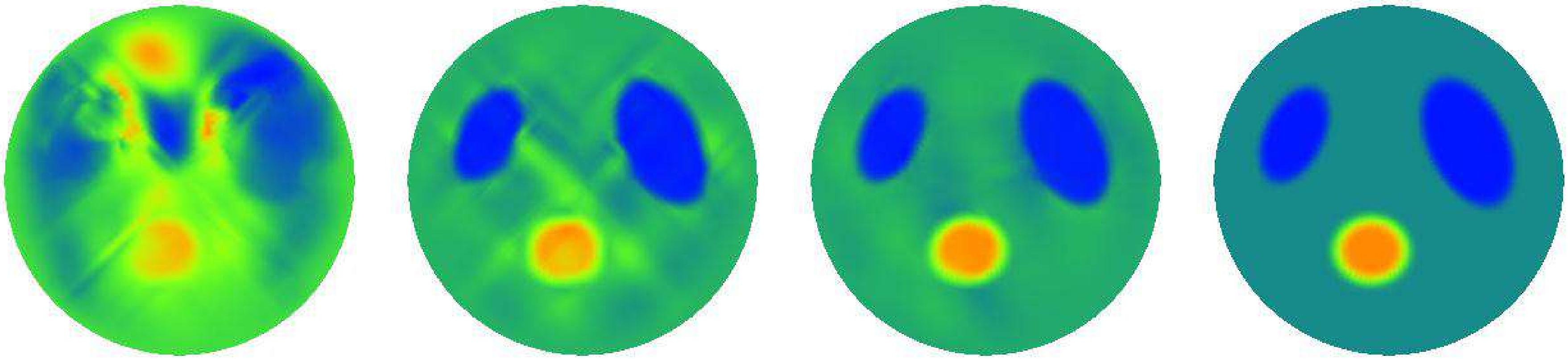
IF A admits a bounded left inverse $(\mathcal{F}')^{-1}(v_0)$, then:

$$v=\mathcal{G}(v) := (\mathcal{F}')^{-1}(v_0)(\mathcal{H}-\mathcal{H}_0)-(\mathcal{F}')^{-1}(v_0)\Big(\mathcal{F}(v_0+v)-\mathcal{F}(v_0)-\mathcal{F}'(v_0)v\Big).$$

$\mathcal{G}(v)$ contraction when $\mathcal{H}-\mathcal{H}_0$ small:

Local uniqueness result for nonlinear HIP.

UMT reconstructions



Reconstruction (Newton iterations based on system $\mathcal{A}^t\mathcal{A}v = \mathcal{A}^t\mathcal{S}$) with: (i) one H; (ii) two H without ellipticity; (iii) two H with ellipticity; (iv) true conductivity.

Calculations by *Kristoffer Hoffmann* (DTU). Theory in joint work with Kristoffer Hoffmann and Kim Knudsen.

Constraints for ellipticity and beyond

• For J **small**, problem may (or may not) be injective with sub-elliptic estimates.

• For J **larger**, problem often is elliptic with optimal stability estimates.

• **Ellipticity** follows from qualitative properties of H_j and u_j, which hold for open set of boundary conditions $\{f_j\}$ (results proved using Complex Geometric Optics (CGO) solutions or Runge approximations).

• Method successfully applied to reconstruction in UMEIT (as above), UMOT:optical parameters (γ, σ) (B. Moskow), Thermo-acoustic tomography (electromagnetic coefficients) (B. Zhou); Photo-acoustic tomography; see also Kuchment-Steinhauer 2012 for a similar elliptic theory for pseudo-differential operators.

• For J **even larger**, more redundant functionals sometimes provide invertible systems by local algebraic manipulations.

Hybrid Problems with very-redundant information

What is to be gained by still increasing J beyond guaranteed ellipticity.

Redundant Internal Functionals with large J

$$-\nabla\cdot\gamma(x)\nabla u_j = 0 \ X, \quad u_j = f_j \ \partial X, \quad H_{ij}(x) = \gamma(x)\nabla u_i\cdot\nabla u_j(x), \ 1\leq i,j\leq J.$$

UMEIT functionals are $H_{ij} = S_i\cdot S_j(x)$ with $S_i(x)=\gamma^{\frac12}\nabla u_i(x)$. Then:

$$\nabla\cdot S_j = -\frac{1}{2}F\cdot S_j, \quad dS_j^\flat = \frac{1}{2}F^\flat\wedge S_j^\flat, \quad 1\leq j\leq J, \qquad F=\nabla(\log\gamma).$$

Strategy: (i) *Eliminate* F and find closed-form equation for $S=(S_1|\ldots|S_n)$.

(ii) Solve a redundant system of ODEs for S.

Step (i) involves *algebraic manipulations* (independent at every point $x\in X$).

Elimination and system of ODEs in UMEIT

Lemma [B.-Bonnetier-Monard-Triki 12; Monard-B. 12].
IF $\inf_{x\in X} \det(S_1(x),\dots,S_n(x)) \geq c_0 > 0$, then with $D(x)=\sqrt{\det H(x)}$,

$$F(x) = \frac{2}{Dn}\sum_{i,j=1}^n \big(\nabla(DH^{ij})\cdot S_i(x)\big)S_j(x), \qquad H^{-1}=(H^{ij}).$$

Moreover, $\nabla\otimes S_j = \sum_{i,k,l,m} H^{ik}(S_k\cdot\nabla S_j)\cdot S_l H^{lm} S_i\otimes S_m$ with

$$2(S_i\cdot\nabla S_j)\cdot S_k = S_i\cdot\nabla H_{jk} - S_j\cdot\nabla H_{ik} + S_k\cdot\nabla H_{ij} - 2F\cdot S_k H_{ij} + 2F\cdot S_j H_{ik}.$$

- By algebraic manipulations (only), we obtain $\boxed{\nabla S = \mathcal{F}(x,S)}$.

Theorem [idem; Capdeboscq et al. SIIS 09 in $n=2$]. There exists open set of f_j for $J=n$ in even dimension and $J=n+1$ in odd dimension such that we have the **global (elliptic) stability** result:

$$\|\gamma-\gamma'\|_{W^{1,\infty}(X)} \leq C\|H-H'\|_{W^{1,\infty}(X)}.$$

Elimination of $F = \nabla(\log\gamma)$

Recall $\nabla\cdot S_j = -F\cdot S_j$ and $dS_j^\flat = F^\flat\wedge S_j^\flat$. Then we introduce

$$X_j^\flat = (-1)^{n+j}\star(S_1^\flat\wedge\ldots\wedge\widehat{S_j^\flat}\wedge\ldots\wedge S_n^\flat) \qquad \text{and find}$$
$$\nabla\cdot X_j = \star d\star X_j^\flat = (-1)^j d(S_1^\flat\wedge\ldots\wedge\widehat{S_j^\flat}\wedge\ldots\wedge S_n^\flat) = (n-1)F\cdot X_j.$$

Now, $X_j\cdot S_k = \delta_{jk}\det S$ so $X_j = DH^{ij}S_i$ with $D = \det H^{\frac12} = \det S$. Thus

$$\begin{aligned}\nabla\cdot X_j &= \nabla(DH^{ij})\cdot S_i + DH^{ij}\nabla\cdot S_i &&= \nabla(DH^{ij})\cdot S_i - DH^{ij}F\cdot S_i \\ &= (n-1)F\cdot(DH^{ij}S_i) &&= (n-1)DH^{ij}F\cdot S_i.\end{aligned}$$

so that [B.-Bonnetier-Monard-Triki'11 & Monard-B.'11]

$$\boxed{F = (H^{ij}F\cdot S_i)S_j = \frac{1}{nD}\Big(\nabla(DH^{ij})\cdot S_i\Big)S_j.}$$

This **eliminates** F to get a closed form equation for $S = (S_1|\ldots|S_n)$. Note that this requires that S form a frame (invertible matrix).

System for frame S

We have $H = S^T S$ and $dS_j^\flat = F^\flat(S) \wedge S_j^\flat$. Not needed: $\nabla \cdot S_j = -F(S) \cdot S_j$.
Can we get $\nabla \otimes S_j = \mathcal{F}_j(S)$ from symmetric and anti-symmetric info.?
This is then a (redundant) system of ODEs.

In Euclidean geometry, the exterior derivative of one forms is

$$dS_i^\flat(S_j, S_k) = S_i \cdot \nabla(S_j \cdot S_k) - S_k \cdot \nabla(S_i \cdot S_k) + [S_i, S_j] \cdot S_k,$$

which gives an expression for the commutator $[S_i, S_j] = S_i \cdot \nabla S_j - S_j \cdot \nabla S_i$.
Also standard expressions for Christoffel symbols give:

$$2(X \cdot \nabla Y) \cdot Z = X \cdot \nabla(Y \cdot Z) + Y \cdot \nabla(X \cdot Z) - Z \cdot \nabla(Y \cdot X) - Y \cdot [X, Z] - Z \cdot [Y, X] + X \cdot [Z, Y].$$

Thus we find for $\nabla \otimes S_j$ in the basis of the vectors S_k:

$$2(S_i \cdot \nabla S_j) \cdot S_k = S_i \cdot \nabla H_{jk} - S_j \cdot \nabla H_{ik} + S_k \cdot \nabla H_{ij} - 2F \cdot S_k H_{ij} + 2F \cdot S_j H_{ik}.$$

Finally

$$\nabla \otimes S_j = \sum_{i,k,l,m} H^{ik} (S_k \cdot \nabla S_j) \cdot S_l H^{lm} S_i \otimes S_m = \mathcal{F}_j(S).$$

Anisotropic conductivities and Calderón problem

Let ϕ be a (sufficiently smooth) diffeomorphism of $\mathbb{R}^n$. Then u solves

$$\nabla \cdot (\gamma \nabla u) = 0$$

if and only if the function $v = u \circ \phi^{-1} = \phi_\star u$ solves

$$\nabla' \cdot (\phi_\star \gamma \nabla' v) = 0, \qquad \phi_\star \gamma(x') := \frac{1}{J_\phi(x)} D\phi^t(x)\ \gamma(x)\ D\phi(x)\Big|_{x=\phi^{-1}(x')}.$$

If ϕ maps X to X and preserves each $x \in \partial X$, then the Dirichlet to Neumann map (**boundary measurements**) satisfies

$$\mathfrak{M}(\gamma) = \mathfrak{M}(\phi_\star \gamma).$$

In other words, we cannot reconstruct γ uniquely from $\mathfrak{M}(\gamma)$. In $n = 2$, this is the only obstruction. In $n \geq 3$, the same holds in the analytic case.

Reconstruction of Anisotropic coefficients

$$\nabla \cdot \gamma \nabla u_i = 0 \ \ X, \quad u_i = f_i \ \ \partial X, \qquad H_{ij} = \gamma \nabla u_i \cdot \nabla u_j, \ \ 1 \le i,j \le I.$$

Define $\gamma = A^2$ and $A = |A|\tilde{A}$ with $\det(\tilde{A}) = 1$. Then for $n = 2$:

Theorem [Monard B. 12] The internal functionals $H = \{H_{ij}\}_{i,j=1}^4$ uniquely determine the tensor $\tilde{A}$ via explicit algebraic equations. Moreover, we have the (still-elliptic) stability estimate

$$\|\tilde{A} - \tilde{A}'\|_{L^\infty(X)} \le C\|H - H'\|_{W^{1,\infty}}.$$

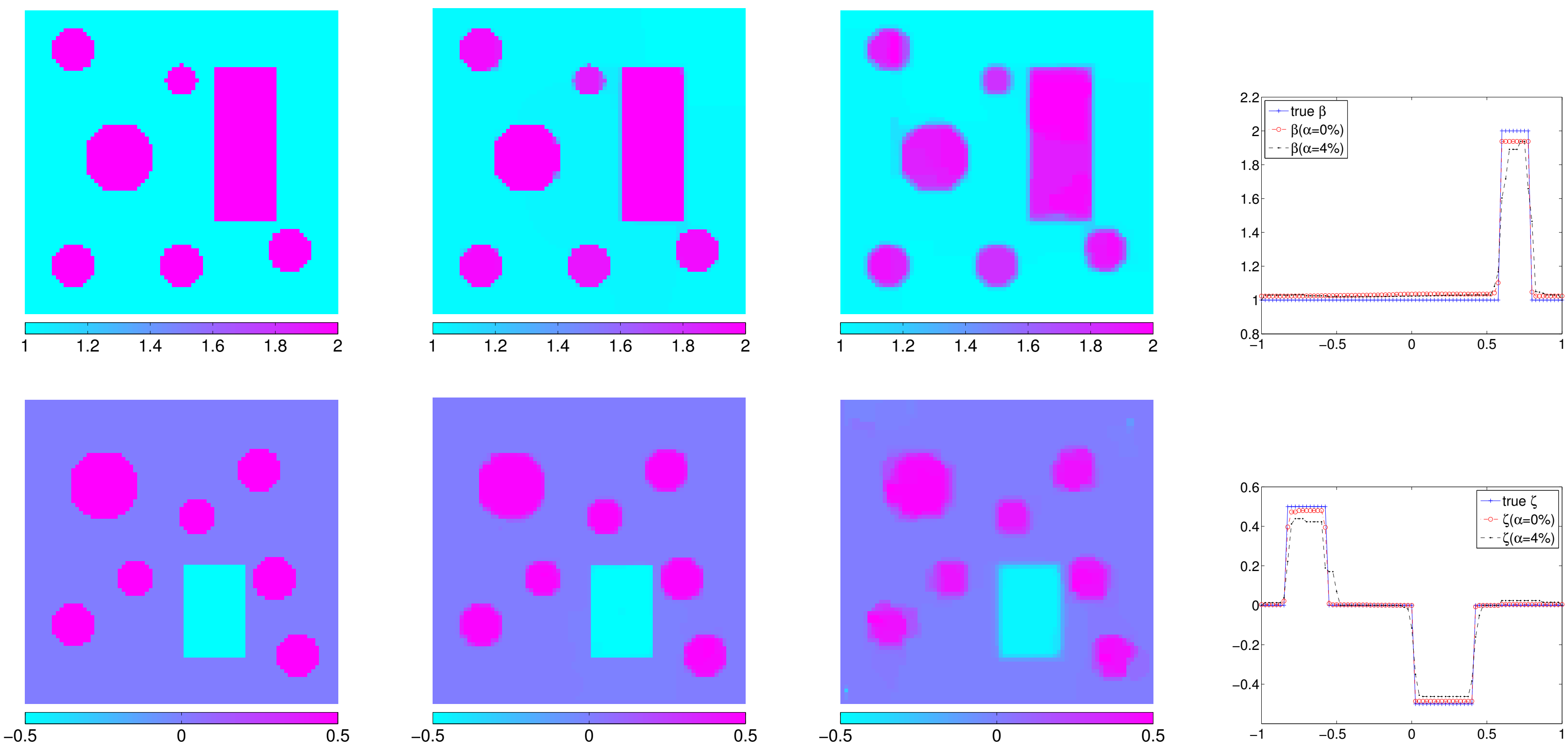
Theorem [Monard B. 12] Let $\tilde{A}$ be known. Then $|A|$ is uniquely determined by $\{H_{ij}\}_{1\le i,j\le 2} \in W^{1,\infty}$. Moreover, we have the (elliptic) estimate

$$\||A| - |A'|\|_{W^{1,\infty}(X)} \le C\|H - H'\|_{W^{1,\infty}}.$$

• Theory applies to higher dimensions and as we saw, to other problems.

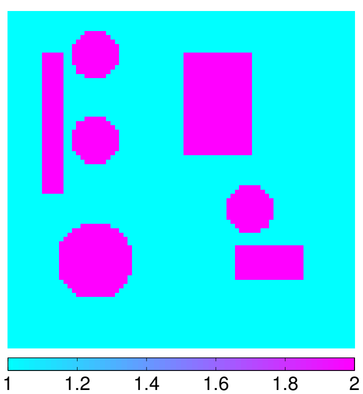
Monard-B. CPDE 2013; B-Uhlmann CPAM 2013; B-Guo-Monard, IP & IPI 2014.

Reconstructions from (4) MR-EIT data

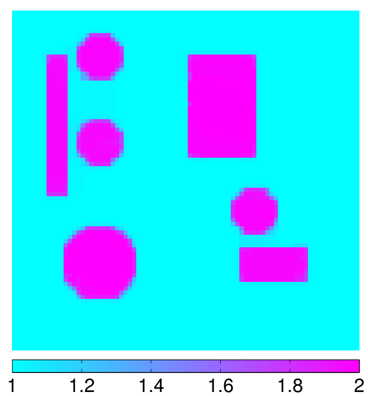


Anisotropy No noise 4% noise + TV Cross section

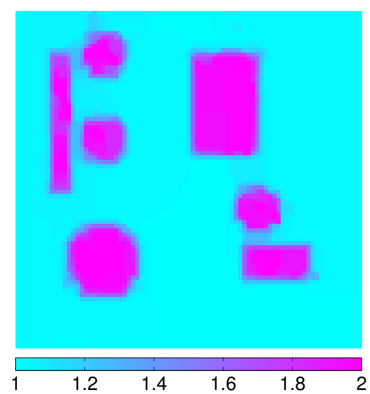
Reconstructions from (4) MR-EIT data



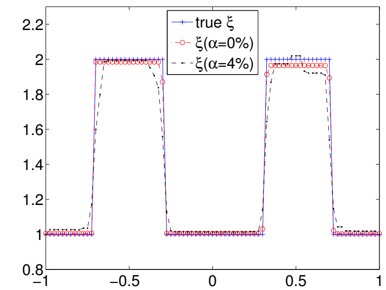
Determinant



No noise

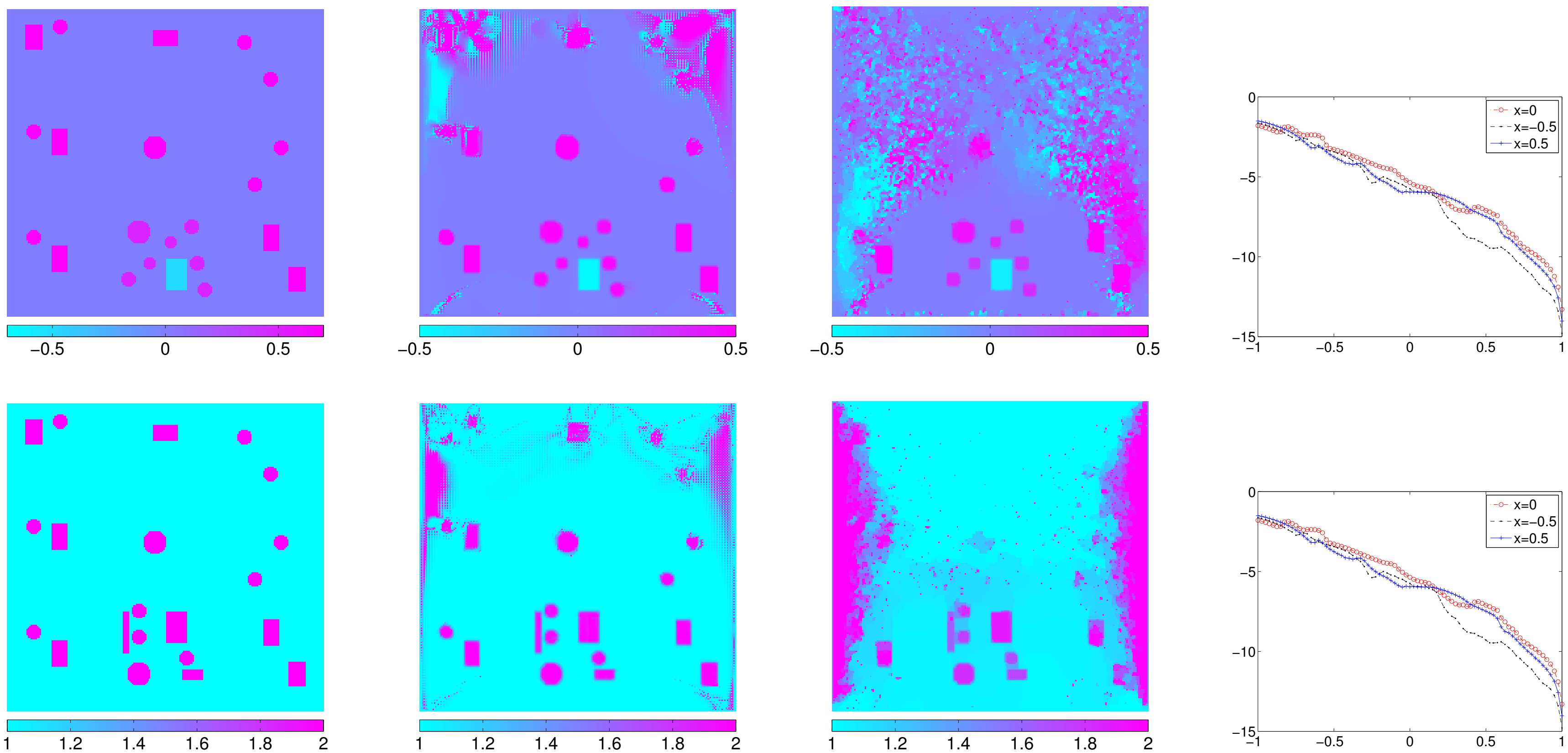


4% noise + TV



Cross section

Reconstructions from (4) bottom illuminations



Coefficient No noise 4% noise + TV Determinant

Independence of ∇u_j not valid close to boundary were $u_j = 0$ is imposed.

Qualitative Properties of Elliptic Solutions

The IFs and the CGOs

Several HIPs require to verify qualitative properties of elliptic solutions:

- the absence of critical points in Photo-acoustics and Elastography
- the hyperbolicity of a given Lorentzian metric in UMOT
- the **linear independence of gradients** of elliptic solutions in UMOT
- the joint ellipticity of quadratic forms in UMEIT

(i) Use **CGO** solutions whenever available: verify the property on unperturbed CGOs (for constant-coefficient equation), by continuity on perturbed CGOs, and then for close-by **illuminations** f_j on ∂X.

(ii) When CGO solutions are not available (anisotropic or complex valued coefficients), construct local solutions (by freezing coefficients) that satisfy such conditions. Then use UCP and the Runge approximation to control such solutions from ∂X.

When qualitative properties fail to hold, stability degrades (Alessandrini et al. QPAT)

Vector fields and complex geometrical optics

• Take $\rho = (\rho_r + i\rho_i) \in \mathbb{C}^n$ with $\rho \cdot \rho = 0$. Then $\Delta e^{\rho\cdot x} = 0$. Let $u_1 = \Re e^{\rho\cdot x}$ and $u_2 = \Im e^{\rho\cdot x}$ so that $\nabla u_1 = e^{\rho_r\cdot x}\Big(\cos(\rho_i\cdot x\rho_r) - \sin(\rho_i\cdot x\rho_i)\Big)$ and $\nabla u_2 = e^{\rho_r\cdot x}\Big(\sin(\rho_i\cdot x\rho_r) + \cos(\rho_i\cdot x\rho_i)\Big)$. We thus find that

$$|\nabla u_1| > 0, \qquad |\nabla u_2| > 0, \qquad \nabla u_1 \cdot \nabla u_2 = 0.$$

• Let $u_\rho(x) = \gamma^{-\frac12} e^{\rho\cdot x}\Big(1 + \psi_\rho(x)\Big)$ solution of $-\nabla\cdot\gamma\nabla u_\rho + \sigma u_\rho = 0$.
Define $q = -\gamma^{-\frac12}\Delta\gamma^{\frac12} - \gamma^{-1}\sigma$.

Theorem[B.-Uhlmann 10]. For q sufficiently smooth and $k \geq 0$, we have

$$|\rho|\|\psi_\rho\|_{H^{\frac{n}{2}+k+\varepsilon}(X)} + \|\psi_\rho\|_{H^{\frac{n}{2}+k+1+\varepsilon}(X)} \leq C\|q\|_{H^{\frac{n}{2}+k+\varepsilon}(X)}.$$

• Thus the perturbed gradient directions $\theta_1 = \widehat{\nabla u_1}$ and $\theta_2 = \widehat{\nabla u_2}$ still satisfy $|\theta_1| > 0$, $|\theta_2| > 0$, and $|\theta_1\cdot\theta_2| \ll 1$ locally so that (θ_1, θ_2) are linearly independent on the bounded domain X of interest.

Existence of critical points

Theorem: Let $X \subset \mathbb{R}^3$ be a bounded Lipschitz domain. Take $g \in C(\partial X) \cap H^{\frac{1}{2}}(\partial X)$. Then there exists a nonempty open set of conductivities $\sigma \in C^\infty(\overline{X})$, $\sigma \geq 1/2$, such that the solution $u \in H^1(X)$ to

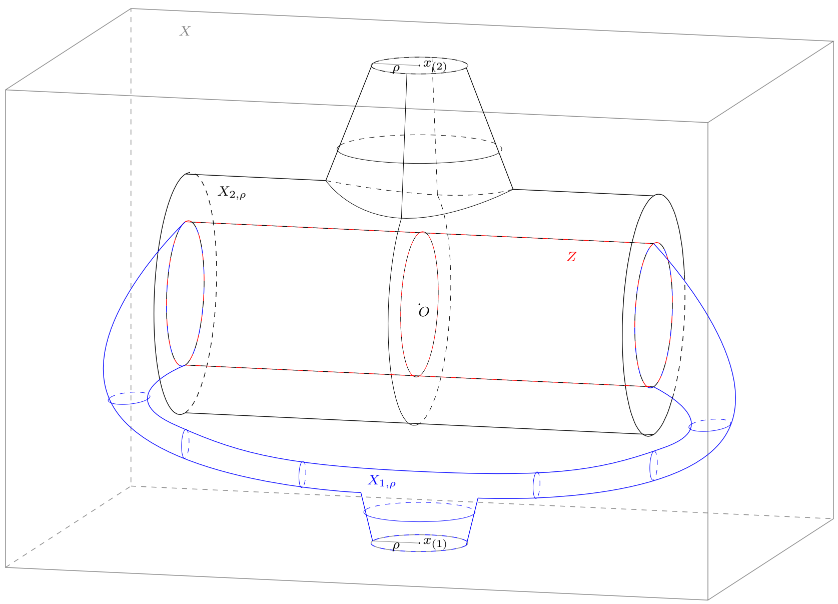
$$-\nabla \cdot \sigma \nabla u = 0 \quad \text{in } X, \qquad u = g \quad \text{on } \partial X$$

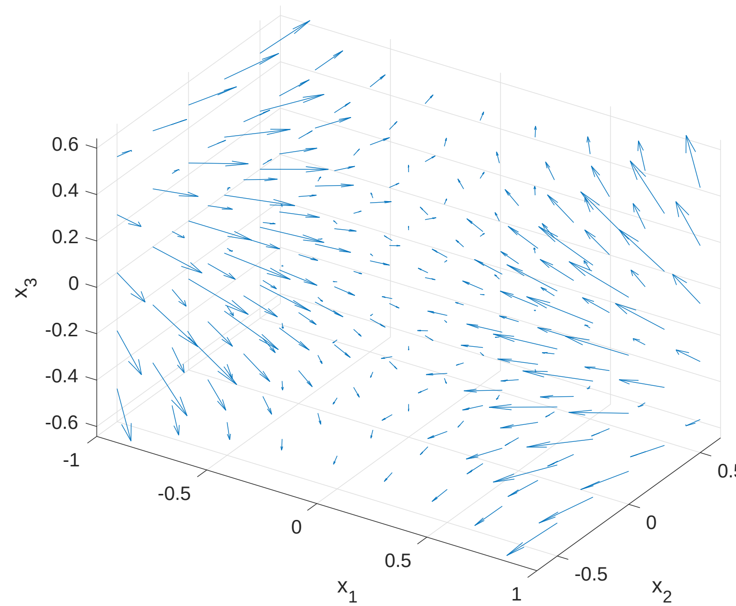
has a critical point in X, namely $\nabla u(x) = 0$ for some $x \in X$ (depending on σ).

In spatial dimension $n = 2$, it is known that the number of critical points (where $\nabla u = 0$) is related to the number of oscillations of the boundary condition independently of the (positive) coefficient σ. The situation is thus very different in dimension $n \geq 3$.

ARMA 2017 (joint with **Giovanni Alberti** and **Michele Di Cristo**).

Geometry and topology





Right: geometry of construction of σ generating $\nabla u = 0$.

Left: local topology of ∇u and topological obstruction to $\nabla u \neq 0$.

Conclusions for Elliptic Hybrid Inverse Problems

• **Hybrid imaging modalities** provide stable inverse problems combining **high resolution** with **high contrast** (though they are Low Signal).

• They often form **systems of nonlinear PDE**, with optimal stability estimate obtained for elliptic (often redundant) systems.

• Additional redundancy may provide algebraic/explicit reconstructions.

• **Tensors** and **Complex-valued** coefficients can be reconstructed to account for *anisotropy* and *dispersion* effects.

• **CGO** solutions and unique continuation properties useful to show existence of *well-chosen* boundary conditions. Such BCs are necessarily somewhat dependent on the (unknown) elliptic coefficients.