On detecting and disclosing hidden acoustic inhomogeneities, by stochastically following random, though monitored, walks

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Outline

- Exterior Boundary value problems
- The stochastic calculus in the service of differential equations
- Stochastic processes and interior boundary value problems as the primitive approach
- On the stochastic representation of the solution of the exterior Dirichlet problem for Helmholtz operator.
- Implementation of two reconstruction methods: Far field and near field inversion

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Exterior elliptic problems

$$
\begin{aligned}
&Lu(x) = 0, \quad x \in D^e = \mathbb{R}^n \backslash \bar{D} \\
&u(x) = f(x), \quad x \in \partial D \\
&\mathcal{M}u(x) = 0, \quad x \in \mathbb{R}^n
\end{aligned}
$$

where L is an elliptic - or better the opposite of an elliptic - differential operator of second order, which in the realm of the present work is expressed generally by the formula $L = \frac{1}{2}\sum_{i,j} a_{ij}(x)\frac{\partial^2}{\partial x_i \partial x_j} + \sum_i b_i(x)\frac{\partial}{\partial x_i} + c_0(x)$,
$f(x)$ represents the Dirichlet data on the surface ∂D, while the function $\mathcal{M}u(x)$ is the more intricate term of the scheme and is constructed via the application of the second Green's identity to the pair of functions consisted of the solution $u(x)$ and the fundamental free space solution $G(x, y)$ corresponding to the operator L.

Helmholtz equation (n=3)

The acoustic scattering field $u(x)\exp(-i\omega t)$ emanated from the interference of an incident time harmonic wave $u^{in}(x,t) = \exp(i(k\hat{k}\cdot x - \omega t))$ with the soft scatterer $\bar{D} \subset \mathbb{R}^n$ satisfies the following boundary value problem ($k = \frac{\omega}{c}$)

$$
\begin{aligned}
&(\Delta + k^2)u(x) = 0, \quad x \in D^e\\
&u(x) = -\exp(ik\hat{k}\cdot x), \quad x \in \partial D\\
&\lim_{r\to\infty} r^{-1}\left(\frac{\partial u(x)}{\partial r} - iku(x)\right) = 0,
\end{aligned}
$$

where $k \neq 0$ is the wave number, the unit vector $\hat{k}$ indicates the direction of the incident wave and ω stands for the angular frequency of the scattering process. Relation $\mathcal{M}u(x) = 0$ leads to the Sommerfeld radiation condition which holds uniformly over all possible direction $\hat{x} = \frac{x}{r}$. This condition not only gives information about the asymptotic behavior of the scattered wave but also incorporates the physical property according to which the whole energy of the scattered wave travels outwards leaving behind the scatterer from whom emanates.

$$u(x) = \frac{1}{|x|}u_\infty(\hat{x};\hat{k},k) + u_1(x), \ \ |x|u_1(x) \to 0, \ \text{as} \ \ |x| \to \infty$$

where we recognize the far-field pattern $u_\infty(\hat{x};\hat{k},k)$ totally characterizing the behavior of the wave field $u(x)$ a few wave-lengths away the scatterer D.

The direct and inverse boundary value problem

- In all cases, the direct exterior boundary value problem consists in the determination of the field $u(x)$ outside D when boundary data (i.e the function f) and geometry (i.e the shape of ∂D) are given. In fact, in most applications, we are interested in determining the remote pattern of this field far away the bounded domain D. For example, in three dimensions, it would be sufficient to determine the far field pattern $u_\infty(\hat{x}; \hat{k})$ if we deal with an application in which we do not have access near the domain D.
- The inverse exterior boundary value problem aims at determining the shape of the surface ∂D when the boundary data is known and the remote pattern is measured. Equivalently, instead of considering as data the measured remote field, it is usual to have at hand the Dirichlet to Neumann (DtN) operator on a sphere - or part of it - surrounding the domain D. In other words a large class of interesting inverse boundary value problem are based on data incorporating both the measured field along with its normal derivative on a given surface belonging to the near field region.

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Stochastic Approach to Deterministic Boundary Value problems

- The most celebrated example is the stochastic solution of the Dirichlet problem involving Laplace operator:
  Given a (reasonable) domain D in $\mathbb{R}^n$ and a continuous function f on the boundary ∂D, find a continuous function u on the closure $\overline{D}$ such that

$$(i)\ \ u = f \ \ on \ \ \partial D,$$

$$(ii)\ \ \Delta u = \sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2} = 0, \ \ in \ \ D.$$

- In 1944, Kakutani proved that the solution could be expressed in terms of a Brownian motion : In fact $u(x),\ x \in D$ is the expected value of f at the first exit point from D of a Brownian motion starting at $x \in D$.
- This was only the beginning. For a large class of second order elliptic differential equations, the corresponding Dirichlet boundary value problem can be solved on the basis of a suitable stochastic process which is a solution of a systematically associated stochastic ordinary differential equation.

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Basics of probability theory

- We consider a probability space consisted in a specific triple $(\Omega, \mathcal{F}, P)$.
  In this structure, Ω is a given set.
  $\mathcal{F}$ is a $\sigma-$algebra on Ω. More precisely, it is a family of subsets of Ω, whose membership obeys to basic rules: The empty set $\emptyset$ is always present in this collection, the complement of every set of Ω can not be absent, while the infinite union of sets participating in $\mathcal{F}$ is also there.
  The third part of the triple is the probability measure $P : \mathcal{F} \longmapsto [0, 1]$, assigning zero value to the empty set, value one to the whole space Ω and being countably additive on disjoint sets.
  $P(F) =$ " The probability that the event $F \in \mathcal{F}$ occurs ".
- If $\mathcal{U}$ is any family of subsets of Ω, then we define the $\sigma-$algebra generated by $\mathcal{U}$ as follows: $\mathcal{H}_{\mathcal{U}} = \cap\{\mathcal{H}; \mathcal{H}\ \sigma-\text{algebra of } \Omega,\ \mathcal{U} \subset \mathcal{H}\}$.
  The most characteristic case is the Borel σ-algebra $\mathcal{B}$ on Ω, generated by the collection of all open sets of a topological space Ω. The elements $B \in \mathcal{B}$ are called Borel sets.
- A function $Y : \Omega \longmapsto \mathbb{R}^n$ is called $\mathcal{F}-$ *measurable* if $Y^{-1}(U) := \{\omega \in \Omega; Y(\omega) \in U\} \in \mathcal{F}$ for all open subsets $U \in \mathbb{R}^n$ (or, equivalently, for all Borel sets $U \in \mathbb{R}^n$).
- If $X : \Omega \longmapsto \mathbb{R}^n$ is any function, then the $\sigma-$algebra $\mathcal{H}_X$ generated by X is the smallest $\sigma-$algebra on Ω containing all the sets $X^{-1}(U)$; $U \in \mathbb{R}^n$ open. In fact, $\mathcal{H}_X = \{X^{-1}(B); B \in \mathcal{B}\}$, where, here, $\mathcal{B}$ is the Borel $\sigma-$algebra on $\mathbb{R}^n$.

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[1] The parameter space T is usually the halfline $[0,\infty)$ but it may also be an interval $[a,b]$, the non-negative integers and even subsets of $\mathbb{R}^m$ for $m\geq 1$.

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Basics of probability theory - Stochastic Processes

- For every specific "*time*" $t \in T$, we have a random variable $\omega \longmapsto X_t(\omega);\ \omega \in \Omega$. For every specific "*experiment*" $\omega \in \Omega$, we have the function $t \longmapsto X_t(\omega);\ t \in T$, which is called a path of X_t.
- Sometimes, it is convenient to write $X(t, \omega)$ instead of $X_t(\omega)$. This is useful in cases it is crucial to have $X(t, \omega)$ jointly measurable in $(t, \omega) \in T \times \Omega$.
- Each $\omega \in \Omega$ can be identified with the function $t \longmapsto X_t(\omega)$ from T into $\mathbb{R}^n$. Thus we may regard Ω as a subset of the space $\widetilde{\Omega} = (\mathbb{R}^n)^T$ of all functions from T into $\mathbb{R}^n$. Then the $\sigma-$algebra $\mathcal{F}$ will contain the $\sigma-$algebra $\mathcal{B}$ generated by the sets of the form $\{\omega;\ \omega(t_1) \in F_1, \ldots, \omega(t_k) \in F_k\},\ \ F_i \in \mathbb{R}^n$ (Borel sets). Therefore, one may also adopt the point of view that a stochastic process is a probability measure P on the measurable space $((\mathbb{R}^n)^T, \mathcal{B})$.
- The finite-dimensional distributions of the process $\{X_t\}_{t\in T}$ are the measures $\mu_{t_1,\ldots,t_k}$ defined on $\mathbb{R}^{nk}$, $k = 1, 2, \ldots$ by $\mu_{t_1,\ldots,t_k}(F_1 \times \ldots \times F_k) = P[X_{t_1} \in F_1, \ldots, X_{t_k} \in F_k];\ t_i \in T$, where $F_1, .., F_k$ are Borel sets in $\mathbb{R}^n$. These distributions determine many (but not all) important properties of the process X.

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- Sometimes, it is convenient to write $X(t,\omega)$ instead of $X_t(\omega)$. This is useful in cases it is crucial to have $X(t,\omega)$ jointly measurable in $(t,\omega) \in T \times \Omega$.
- Each $\omega \in \Omega$ can be identified with the function $t \longmapsto X_t(\omega)$ from T into $\mathbb{R}^n$. Thus we may regard Ω as a subset of the space $\widetilde{\Omega} = (\mathbb{R}^n)^T$ of all functions from T into $\mathbb{R}^n$. Then the $\sigma-$algebra $\mathcal{F}$ will contain the $\sigma-$algebra $\mathcal{B}$ generated by the sets of the form $\{\omega;\ \omega(t_1) \in F_1, \ldots, \omega(t_k) \in F_k\},\ F_i \in \mathbb{R}^n$ (Borel sets). Therefore, one may also adopt the point of view that a stochastic process is a probability measure P on the measurable space $((\mathbb{R}^n)^T, \mathcal{B})$.
- The finite-dimensional distributions of the process $\{X_t\}_{t\in T}$ are the measures $\mu_{t_1,\ldots,t_k}$ defined on $\mathbb{R}^{nk}$, $k = 1, 2, \ldots$ by $\mu_{t_1,\ldots,t_k}(F_1 \times \ldots \times F_k) = P[X_{t_1} \in F_1, \ldots, X_{t_k} \in F_k];\ t_i \in T$, where $F_1, .., F_k$ are Borel sets in $\mathbb{R}^n$. These distributions determine many (but not all) important properties of the process X.

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- The finite-dimensional <u>distributions</u> of the process $\{X_t\}_{t \in T}$ are the measures $\mu_{t_1,\ldots,t_k}$ defined on $\mathbb{R}^{nk}$, $k = 1, 2, \ldots$ by $\mu_{t_1,\ldots,t_k}(F_1 \times \ldots \times F_k) = P[X_{t_1} \in F_1, ..., X_{t_k} \in F_k];\ t_i \in T$, where $F_1, .., F_k$ are Borel sets in $\mathbb{R}^n$. These distributions determine many (but not all) important properties of the process X.

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- Conversely, given a family of $\nu_{t_1,\ldots,t_k};\ k \in \mathbb{N},\ t_i \in T$ of probability measures on $\mathbb{R}^{nk}$, it is important to be in position to construct a stochastic process $Y = \{Y_t\}_{t \in T}$ having $\nu_{t_1,\ldots,t_k}$ as its finite dimensional distributions.
- One of Kolmogorov's famous theorems states the following result.

Theorem

For all $t_1, \ldots, t_k \in T,\ k \in \mathbb{N}$, let $\nu_{t_1,\ldots,t_k}$ be probability measures on $\mathbb{R}^{nk}$ s.t. $\nu_{t_{\sigma(1)},\ldots,t_{\sigma(k)}}(F_1 \times \ldots \times F_k) = \nu_{t_1,\ldots,t_k}(F_{\sigma^{-1}(1)} \times \ldots \times F_{\sigma^{-1}(k)})$ for all permutations σ on $\{1, 2, \ldots, k\}$ and

$$\nu_{t_1,\ldots,t_k}(F_1 \times \ldots \times F_k) = \nu_{t_1,\ldots,t_k,t_{k+1},\ldots,t_{k+m}}(F_1 \times \ldots \times F_k \times \underbrace{\mathbb{R}^n \times \ldots \times \mathbb{R}^n}_{m \text{ factors}}) \text{ for all } m \in \mathbb{N}.$$

Then there exists a probability space $(\Omega, \mathcal{F}, P)$ and a stochastic process $\{X_t\}$ on Ω, $X_t : \Omega \longmapsto \mathbb{R}^n$, s.t. $\nu_{t_1,\ldots,t_k}(F_1 \times \ldots \times F_k) = P[X_{t_1} \in F_1, \ldots, X_{t_k} \in F_k]$.

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Basics of probability theory-Brownian motion

- We consider a specific family of probability measures $\nu_{t_1,\dots t_k}$, consistent with consecutive gaussian transitions starting from an initial point $x \in \mathbb{R}^n$:

$$p(t,x,y) = (2\pi t)^{-n/2} \exp\left(-\frac{|x-y|^2}{2t}\right), \quad y \in \mathbb{R}^n, \ t > 0.$$

More precisely, if $0 \le t_1 < t_2 < \dots < t_k$, we define a measure $\nu_{t_1,\dots t_k}$ on $\mathbb{R}^{nk}$ by

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Kolmogorov's theorem establishes the existence of a probability space $(\Omega, \mathcal{F}, P^x)$ and the well known accompanying Brownian motion B_t (starting at x) satisfying

$$\nu_{t_1,\dots t_k}(F_1 \times F_2 \times \dots \times F_k) = P^x[B_{t_1} \in F_1, \dots, B_{t_k} \in F_k];$$

- The canonical Brownian motion is the space $C([0,\infty), \mathbb{R}^n)$ equipped with certain probability measures P^x as described above. Under the law P^x, $E^x[B_t] = x$ for all $t \ge 0$, $E^x[(B_t - B_s)^2] = n(t-s)$, for $t \ge s$ and B_t has independent increments B_{t_1}, $B_{t_2} - B_{t_1}$,...,$B_{t_k} - B_{t_{k-1}}$ for all $0 \le t_1 < t_2 < \dots < t_k$.
- In addition we define as $\mathcal{F}_t = \mathcal{F}_t^{(n)}$ to be the σ-algebra generated by the random variables $\{B_i(s)\}_{1 \le i \le n; 0 \le s \le t}$. We can think $\mathcal{F}_t$ as "the history of B_s up to time t". By construction, the Brownian motion is of course $\{\mathcal{F}_t\}$-adapted. The *first exit time* $\tau_U := \inf\{t > 0; B_t \notin U\}$ of a Brownian motion from an open set $U \subset R^n$ is a stopping time w.r.t $\{\mathcal{F}_t\}$. (i.e. $\{\omega; \tau(\omega) \le t\} \in \mathcal{F}_t$, for all $t \ge 0$).

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Basics of stochastic differential equations

- We are interested in solving differential equations with noise

  $\frac{dX_t}{dt} = b(t, X_t) + \sigma(t, X_t) \cdot$ "noise"
- It is reasonable to look for some stochastic process W_t, to represent the noise term (the 1-dimensional case).

  $\frac{dX_t}{dt} = b(t, X_t) + \sigma(t, X_t) \cdot W_t$
- Inspired by applications, it seems necessary to ask (a) W_{t_1}, W_{t_2} being independent for $t_1 \neq t_2$, (b) some kind of stationarity imposing that the joint distribution of $\{W_{t_1+t}, \dots, W_{t_k+t}\}$ is independent of t and finally (c) that $E[W_t] = 0$.
- Unfortunately, it turns out that no reasonable stochastic process exists satisfying these assumptions. Actually, such a W_t cannot have continuous paths.
- Nevertheless, it is feasible to represent W_t as a generalized stochastic process, constructed as a probability measure on the space of tempered distributions $\mathcal{S}'$, called the white noise process.
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- We replace $W_k \Delta t_k = \Delta V_k = V_{t_{k+1}} - V_{t_k}$, where $\{V_t\}_{t \geq 0}$ is a suitable stochastic process. The assumptions $(a), (b)$ and (c) imply that V_t should have *stationary independent increments with mean* 0.
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Basics of stochastic differential equations

- Let $0 = t_0 < t_1 < ... < t_m = t$ and consider the discrete analogue $X_{k+1} - X_k = b(t_k, X_k)\Delta t_k + \sigma(t_k, X_k) W_k \Delta t_k$ with $X_j = X(t_j),\ W_k = W_{t_k},\ \Delta t_k = t_{k+1} - t_k$.
- We replace $W_k \Delta t_k = \Delta V_k = V_{t_{k+1}} - V_{t_k}$, where $\{V_t\}_{t\geq 0}$ is a suitable stochastic process. The assumptions $(a), (b)$ and (c) imply that V_t should have *stationary independent increments with mean* 0.
- It turns out that the only such process with continuous paths is the Brownian[2] motion B_t. We put $V_t = B_t$ and obtain
- $$X_k = X_0 + \sum_{j=0}^{k-1} b(t_j, X_j)\Delta t_j + \sum_{j=0}^{k-1} \sigma(t_j, X_j)\Delta B_j.$$
- $$X_t = X_0 + \int_0^t b(s, X_s) ds + \int_0^t \sigma(s, X_s) dB_s$$
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- $f(t,\omega)\in\mathcal{V}=\mathcal{V}(S,T)$ where $\mathcal{V}(S,T)$ contains functions such that $(t,\omega)\longmapsto f(t,\omega)$ is $\mathcal{B}\times\mathcal{F}-$measurable, $f(t,\omega)$ is $\mathcal{F}-$adapted and $E[\int_S^T f(t,\omega)^2 dt] < \infty$.
- A function $\phi \in \mathcal{V}$ is called elementary if $\phi(t,\omega) = \sum_j e_j(\omega)\mathcal{X}_{[t_j,t_{j+1})}(t)$. Then we define
- $\int_S^T \phi(t,\omega)dB_t(\omega) = \sum_j e_j(\omega)[B_{t_{j+1}} - B_{t_j}](\omega)$
- The Itô isometry $E\left[\left(\int_s^T \phi(t,\omega)dB_t(\omega)\right)^2\right] = E\left[\int_s^T \phi^2(t,\omega)dt\right]$ is valid for all the elementary functions.
- For every $f \in \mathcal{V}$, we define the Itô integral of f (from S to T) as $\int_S^T f(t,\omega)dB_t(\omega) = \lim_{n\to\infty} \int_S^T \phi_n(t,\omega)dB_t(\omega)$ (limit in $L^2(P)$) where ϕ_n is a sequence of elementary functions such that $E\left[\int_S^T (f(t,\omega) - \phi_n(t,\omega))^2 dt\right] \underset{n\to\infty}{\rightarrow} 0$.
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Multi-dimensional stochastic ordinary differential equations

In the core of the present work lie the stochastic differential equations of the type

$$dX_t = b(t, X_t)dt + \sigma(t, X_t)dB_t, \quad 0 \leq t \leq T, \;\; X_0 = Z.$$

In the equation above, $T > 0$ while $b(.,.) : [0, \infty) \times \mathbb{R}^n \to \mathbb{R}^n$ and $\sigma(.,.) : [0, \infty) \times \mathbb{R}^n \to \mathbb{R}^{n \times m}$ are measurable functions. The Brownian motion is m-dimensional while the initial state random variable Z is independent of the σ-algebra $\mathcal{F}_\infty^{(m)}$ generated by the Brownian motion at all times. It is proved that under certain strict conditions on b and σ, the stochastic differential equation has a unique t-continuous solution $X_t(\omega)$, which is adapted to the filtration (increasing family) $\mathcal{F}_t^Z$ generated by Z and $B_s; s \leq t$. In addition $E[\int_0^T |X_t|^2 dt < \infty]$. We may integrate obtaining

$$X_t = X_0 + \int_0^T b(t, X_t)dt + \int_0^T \sigma(t, X_t)dB_t$$

where we recognize the Itô integral $\int_0^T \sigma(t, X_t)dB_t$, which is well defined given that the solution X_t involving in the integrand is $\mathcal{F}_t^Z$-adapted. The strict conditions mentioned above impose at most linear growth and Lipschitz behavior of the coefficients both in terms of the second spatial argument, uniformly over time.

The multi-dimensional Itô formula.

It is an issue of great importance to investigate the behavior of composite functions of the form $F(t,\omega) = f(t, X_t) = f(t, X(t))$, where $f(t,x) = (f_1(t,x), f_2(t,x), ..., f_p(t,x))$ is a C^2 map from $[0,\infty) \times \mathbb{R}^n$ into $\mathbb{R}^p$. The method for this effort is provided by the well known multi-dimensional Itô formula, according to which $F(t,\omega)$ is again an Itô process with components F_k, $k = 1, 2, ..., p$, satisfying

$$dF_k = df_k = \frac{\partial f_k}{\partial t}(t,X)dt + \sum_i \frac{\partial f_k}{\partial x^i}(t,X)dX^i + \frac{1}{2}\sum_{i,j}\frac{\partial^2 f_k}{\partial x^i \partial x^j}(t,X)dX^i dX^j$$

where the relations $dB^i dB^j = \delta_{ij}dt$, $dB^i dt = dt dB^i = dt \cdot dt = 0$ span the calculus of products between infinitesimals.

On the basis of the differential scheme $dX_t = b(t,X_t)dt + \sigma(t,X_t)dB_t$, we find that

$$f_k(t,X_t) = f_k(0,x) + \int_0^t \frac{\partial f_k}{\partial s}(s,X_s)ds + \int_0^t \sum_i b_i(s,X_s)\frac{\partial f_k}{\partial x^i}(s,X_s)ds + \int_0^t \frac{1}{2}\sum_{i,j}(\sigma\sigma^T)_{i,j}\frac{\partial^2 f_k}{\partial x^i \partial x^j}(s,X_s)ds$$
$$+ \int_0^t \sum_i \frac{\partial f_k}{\partial x^i}\sigma_{i,j}(s,X_s)dB^j(s)$$

In vector form

$$f(t,X_t) = f(0,x) + \int_0^t \frac{\partial f}{\partial s}(s,X_s)ds + \int_0^t b(s,X_s)\cdot\nabla f(s,X_s)ds$$
$$+\frac{1}{2}\int_0^t (\sigma\sigma^T)(s,X_s) : \nabla\nabla f(s,X_s)ds + \int_0^t (\nabla f)^T(s,X_s)\cdot\sigma(s,X_s)\cdot dB(s).$$

The infinitesimal generator of an Itô diffusion

- A (time-homogeneous) Itô diffusion is a stochastic process $X_t(\omega) = X(t,\omega) : [0,\infty) \longmapsto \mathbb{R}^n$, satisfying a stochastic differential equation of the form $dX_t = b(X_t)dt + \sigma(X_t)dB_t$.
- The integral form of the multi-dimensional Itô formula simplifies to
  $f(X_t) = f(x) + \int_0^t b(X_s)\cdot\nabla f(X_s)ds + \frac{1}{2}\int_0^t (\sigma\sigma^T)(X_s):\nabla\nabla f(X_s)ds$
  $+ \int_0^t (\nabla f)^T(X_s)\cdot\sigma(X_s)\cdot dB(s)$
- We take expectation value
  $E^x[f(X_t)] = f(x) + E^x\left[\int_0^t b(X_s)\cdot\nabla f(X_s)ds\right] + E^x\left[\frac{1}{2}\int_0^t (\sigma\sigma^T)(X_s):\nabla\nabla f(X_s)ds\right]$
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- For every Itô diffusion X_t in $\mathbb{R}^n$, the infinitesimal *generator* A is defined by $Af(x) = lim_{t\downarrow 0}\frac{E^x[f(X_t)]-f(x)}{t}$, $x \in \mathbb{R}^n$ and has a domain D_A including $C_0^2(\mathbb{R}^n)$. More precisely, every $f \in C_0^2(R^n)$ belongs to D_A and satisfies
  $Af(x) = \sum_i b_i(x)\frac{\partial f}{\partial x_i} + \frac{1}{2}\sum_{i,j}(\sigma\sigma^T)_{i,j}(x)\frac{\partial^2 f}{\partial x_i \partial x_j}$.
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The Dynkin's formula.

- Let $f \in C_0^2(\mathbb{R}^n)$ and suppose that τ is a stopping time with $E^x[\tau] < \infty$. Then

$$E^x[f(X_\tau)] = f(x) + E^x\left[\int_0^\tau Af(X_s)ds\right]$$

- The existence of a compact support for the functions f is not necessary if τ is the first exit time of a bounded set.
- ...recall...that

$$dX_t = b(X_t)dt + \sigma(X_t)dB_t, \quad 0 \le t \le T, \;\; X_0 = x$$

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Stochastic representation of the solution of the interior Dirichlet problem.

- Let D be a bounded domain in $\mathbb{R}^n$ and ϕ a bounded function on ∂D, while $u \in C^2(D)$ is supposed to be a solution of the boundary value problem

$$Au(x) := \frac{1}{2}\sum_{i,j} a_{ij}(x)\frac{\partial^2 u(x)}{\partial x_i \partial x_j} + \sum_i b_i(x)\frac{\partial u(x)}{\partial x_i} = 0, \quad x \in D$$

$$\lim_{\substack{x \to y \\ x \in D}} u(x) = \phi(y), \quad y \in \partial D$$

- Exploiting Dynkin's formula, we obtain the representation

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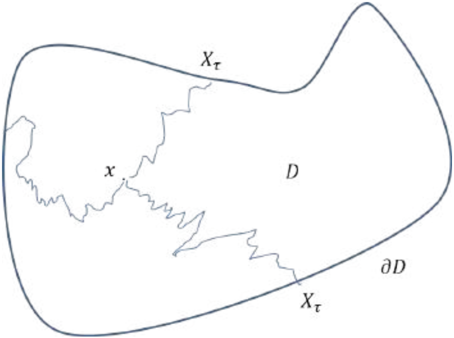
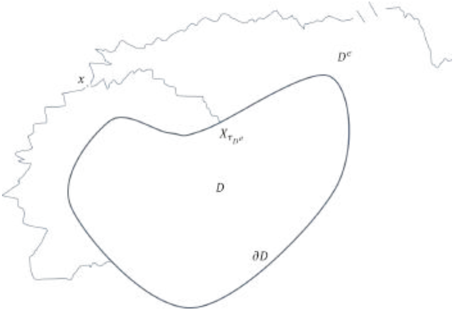
....A nice local method...



$$\Delta u(x) = 0, \quad x \in D \text{ and } u(x) = \phi(x), \; x \in \partial D \Rightarrow u(x) = E^x[\phi(X_{\tau_D})], \quad x \in D$$

Remark that in our case of interest ($A = \frac{1}{2}\Delta$) ...$X_t = B_t$!

What about exterior domains?



The Brownian motion itself is well known to be recurrent in $\mathbb{R}^2$ (i.e. $P^x(\tau_D < \infty) = 1$) but transient in $\mathbb{R}^3$ (i.e. $P^x(\tau_D < \infty) < 1$). So in $\mathbb{R}^3$, a pure Brownian motion could ramble endlessly without hitting the boundary. However, the drift term is often responsible to guide the process toward the exit from the set D^e (or the entrance to the bounded component D).

Helmholtz equation in exterior domains

- $$(\Delta + k^2)u(x) = 0, \quad x \in D^e$$
  $$u(x) = -\exp(ik\hat{k}\cdot x), \quad x \in \partial D$$
  $$\lim_{r\to\infty} r^{-1}\left(\frac{\partial u(x)}{\partial r} - iku(x)\right) = 0,$$

- $$E^x[u(X_\tau)\chi_{\{\tau<\infty\}}] = E^x[u(B_\tau)\chi_{\{\tau<\infty\}}] = u(x) + E^x\left[\int_0^\tau Au(B_s)ds\right] \Rightarrow$$
  $$u(x) = -E^x[\exp(ik\hat{k}\cdot B_\tau)\chi_{\{\tau<\infty\}}] + \frac{k^2}{2}E^x\left[\int_0^\tau u(B_s)ds\right],$$

- However, the validity of this formula requires that $E^x\left[\int_0^\tau |u(B_s)|ds\right] < \infty$, which is strongly ambiguous since u has no compact support and the life time variable is not controllable. In addition, the Monte-Carlo simulation would be very slow since a part of trajectories could ramble for a long time before hitting the boundary or just making eternal loops inside the exterior space D^e. Even if these drawbacks were bypassed, the implication of the integral term is not desirable since it involves the values of the field along several paths and actually necessitates the enrichment of data over a large part of the exterior space, fact which is unrealizable. The information is restricted on the surface of the scatterer (boundary condition) and on the data surface where measurements are gathered.

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Helmholtz equation versus the modified Helmholtz equation

- $$(\Delta - k^2)u(x) = 0, \quad x \in D^e, \ (k > 0)$$
  $$u(x) = f(x), \quad x \in \partial D$$
  $$\lim_{r\to\infty} r^{-1}\left(\frac{\partial u(x)}{\partial r} + ku(x)\right) = 0,$$

- Feynman-Kac formula gives (by ...killing a diffusion)

$$u(x) = E^x\left[f(B_\tau)\exp(-\int_0^\tau k^2 ds)\ \chi_{\{\tau<\infty\}}\right] = E^x\left[f(B_\tau)\exp(-k^2\tau)\ \chi_{\{\tau<\infty\}}\right]$$

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Helmholtz equation via Liouville representation

- Adopt a solution in the Liouville product form : $u(x) = \phi(x)\exp(-ikS(x))$. Then, Helmholtz equation results in
- eikonical equation from ray optics $|\nabla S|^2 = k^2$. It can be treated as a particular case of the Hamilton-Jacobi equation of classical mechanics
- extended transport equation $\frac{i}{2}\Delta\phi(x) + \nabla S(x)\cdot\nabla\phi(x) + \frac{1}{2}(\Delta S(x))\phi(x) = 0,$ $\phi|_{\partial D} = f(x)\exp(iS(x)) = -\exp(ik\hat{k}\cdot x)\exp(iS(x))$
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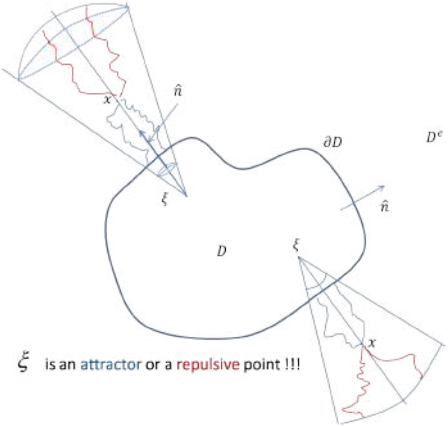
The construction of conditioned stochastic processes for Helmholtz operator. Doob's $h-$transform



$$dY_t = \frac{\nabla h_{m,l}(Y_t)}{h_{m,l}(Y_t)} dt + dB_t, \;\; 0 \leq t \leq T, \;\; Y_0 = x - \xi, \;\;\; Y_t = X_t - \xi, \;\; m \in \mathbb{N}^*, \;\; l = \underline{1, 2}$$

- Let $x \in D^e$ be once again the initial point of the stochastic process under construction. We consider the unit vector $\hat{n}_{x,\xi} := \frac{x-\xi}{|x-\xi|} = \frac{y}{|y|}$. For simplicity we denote $\hat{n}_{x,\xi}$ as $\hat{n}$ since the points x, ξ are assumed as fixed parameters, though the same procedure might be profitable to be applied for several pairs (x, ξ). We introduce now two sets of functions belonging to the kernel of Helmholtz operator. More precisely, evoking the well known Legendre polynomial functions $P_m(\cos\theta), \theta \in [0, \pi]$ and the spherical Bessel (j_m) and Neumann (y_m) functions, we introduce two families ($l = 1, 2$) of eigensolutions of Laplace operator:

$$h_{m,l}(y; k) = P_m(\hat{n} \cdot y/|y|)\mathcal{Q}_{m,l}(k|y|), \;\; m = 0, 1, 2, ...,$$

  where $\mathcal{Q}_{m,1}(k|y|) = j_m(k|y|)$ and $\mathcal{Q}_{m,2}(k|y|) = y_m(k|y|)$. For simplicity, we suppress the dependence on the wavenumber denoting $h_{m,l}(y) = h_{m,l}(y; k)$.
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- $h_{m,1}(Y_t) = P_m(\hat{n} \cdot Y_t/|Y_t|)\mathcal{Q}_{m,1}(k|Y_t|) = P_m(\cos(\Theta_t))j_m(k|Y_t|)$
  Exploiting the recurrence relation of spherical Bessel functions, we find that

$$dY_t = \mathcal{H}_m(k|Y_t|)\frac{Y_t}{|Y_t|^2}dt - \frac{\sin(\Theta_t)P_m'(\cos(\Theta_t))}{|Y_t|P_m(\cos(\Theta_t))}\hat{\Theta}_t dt + dB_t,$$

with

$$\mathcal{H}_m(\lambda) := \frac{\lambda j_{m-1}(\lambda)}{j_m(\lambda)} - (m+1), \quad \lambda > 0.$$

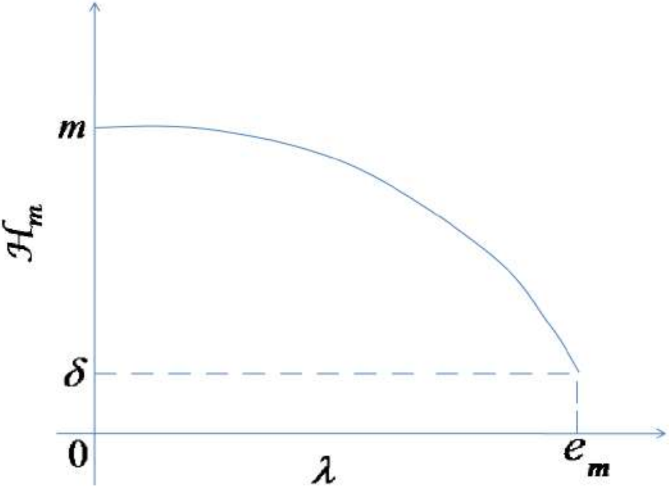

Figure: The decreasing positive function $\mathcal{H}_m(\lambda)$

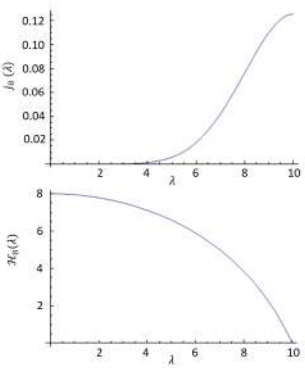

Figure: The decreasing positive function $\mathcal{H}_8(\lambda)$ and the subsequent spherical Bessel function $j_8(\lambda)$.

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$$\tilde{D}^e_{m,\epsilon}(\xi) = D^e \cap \{\xi + y : y \in R^3 \text{ with } \arccos\left(\hat{n}_{x,\xi} \cdot \frac{y}{|y|}\right) \in [0, \theta_{m,1} - \epsilon)\} \cap \{z \in \mathbb{R}^n : \eta < |z - \xi| < L\}$$
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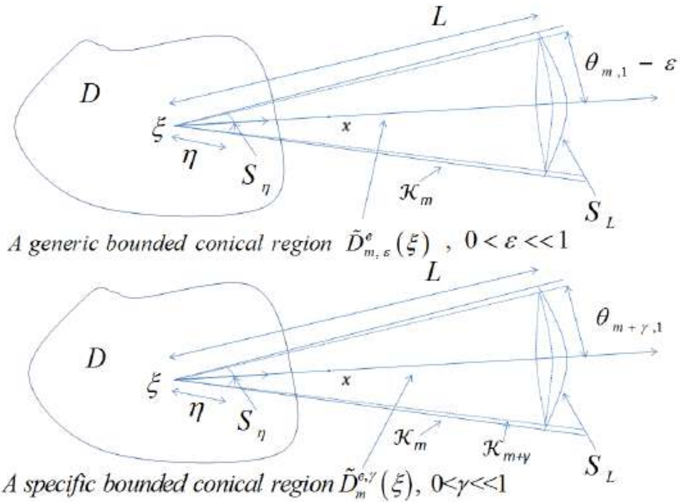

Figure: Inserting cups and interior conical "cushioning" to guarantee regular driving terms.

Parameter m defining the Spherical Bessel j_m	**The maximal height of the cone $\frac{em}{k} = \frac{em}{2\pi}\lambda$**
3	$\frac{\lambda}{2}$
4	$\frac{3\lambda}{4}$
5	λ
8	$\frac{3\lambda}{2}$
11	2λ
29	5λ

Table: Some characteristic cones for several values of the parameter m.

- Apply Dynkin's formula to the field $w(x) = \frac{u(x)}{h_{m,1}(x-\xi)}$
- $$E^x\left[\frac{u(X_\tau)}{h_{m,1}(Y_\tau)}\right] = \frac{u(x)}{h_{m,1}(x-\xi)} + E^x\left[\int_0^\tau A\left(\frac{u(X_s)}{h_{m,1}(Y_s)}\right) ds\right].$$

  Denoting $h = h_{m,1}$, we find that
- $$A\left(\frac{u}{h}\right) = \frac{\nabla h}{h}\cdot\nabla\left(\frac{u}{h}\right) + \frac{1}{2}\Delta\left(\frac{u}{h}\right) = \frac{h\Delta\left(\frac{u}{h}\right) + 2\nabla h\cdot\nabla\left(\frac{u}{h}\right) + \left(\frac{u}{h}\right)\Delta h - \left(\frac{u}{h}\right)\Delta h}{2h} = \frac{\Delta u + k^2 u}{2h} = 0.$$
- $$\frac{u(x)}{j_m(k|x-\xi|)} = E^x\left[\frac{u(X_\tau)}{h_{m,1}(Y_\tau)}\right] \Rightarrow u(x) = j_m(k|x-\xi|)E^x\left[\frac{u(X_\tau)}{h_{m,1}(Y_\tau)}\right].$$

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- $$E^x\left[\frac{u(X_\tau)}{h_{m,1}(Y_\tau)}\right] = \frac{u(x)}{h_{m,1}(x-\xi)} + E^x\left[\int_0^\tau A\left(\frac{u(X_s)}{h_{m,1}(Y_s)}\right)ds\right].$$

  Denoting $h = h_{m,1}$, we find that
- $$A\left(\frac{u}{h}\right) = \frac{\nabla h}{h}\cdot\nabla\left(\frac{u}{h}\right) + \frac{1}{2}\Delta\left(\frac{u}{h}\right) = \frac{h\Delta\left(\frac{u}{h}\right) + 2\nabla h\cdot\nabla\left(\frac{u}{h}\right) + \left(\frac{u}{h}\right)\Delta h - \left(\frac{u}{h}\right)\Delta h}{2h} = \frac{\Delta u + k^2 u}{2h} = 0.$$
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Stochastic representation of the scattered field (1)

- $$u(x) = \frac{j_m(k|x-\xi|)}{j_m(e_m)} E^x_{\text{ext}}\left[\frac{u(X_\tau)}{P_m(\cos(\Theta_\tau))}\right] + \frac{j_m(k|x-\xi|)}{j_m(k\eta)} E^x_{\text{int}}\left[\frac{u(X_\tau)}{P_m(\cos(\Theta_\tau))}\right]$$
  tends to 0 *as* γ *goes to* 0
  $$+ \cancel{\frac{j_m(k|x-\xi|)}{P_m(\cos(\theta_{m+\gamma,1}))} E^x_{\text{lat}}\left[\frac{u(X_\tau)}{j_m(k|Y_\tau|)}\right]}, \; x \in \tilde{D}^{e,\gamma}_m(\xi) \tag{1}$$

  since...

- $$P^x\left(\left\{X_\tau \in \partial\tilde{D}^{e,\gamma}_m(\xi)\right\} \cap \{\Theta_\tau = \theta_{m+\gamma,1}\} \cap \{\tau < T\}\right) \le e^{\frac{k^2\zeta}{2}T}\left(\frac{j_m(e_m)}{j_m(k|x-\xi|)}\right)^\zeta (P_m(\cos(\theta_{m+\gamma,1})))^\zeta,$$

  for any real $\zeta > 1$

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  for any real $\zeta > 1$

Proposition

If the domain of the stochastic process is $\tilde{D}^{e}_{m,\epsilon}(\xi)$ and τ is the first exit time from this domain, then the probability of escaping from the lateral surface, instead of the cups, converges to zero as the parameter ϵ tends to zero:

$$\lim_{\epsilon \to 0} P^{x}\left(\left\{X_{\tau} \in \partial \tilde{D}^{e}_{m,\epsilon}(\xi)\right\} \cap \{\Theta_{\tau} = \theta_{m,1} - \epsilon\}\right) = 0.$$

Proposition

If the domain of the stochastic process is $\tilde{D}^{e,\gamma}_{m}(\xi)$ and τ is the first exit time from this domain, then the probability of escaping from the lateral surface, instead of the cups, in finite time T has the estimate

$$P^{x}\left(\left\{X_{\tau} \in \partial \tilde{D}^{e,\gamma}_{m}(\xi)\right\} \cap \{\Theta_{\tau} = \theta_{m+\gamma,1}\} \cap \{\tau < T\}\right) \leq e^{\frac{k^2\zeta}{2}T}\left(\frac{j_m(e_m)}{j_m(k|x-\xi|)}\right)^{\zeta} P^{\zeta}_{m}(\cos(\theta_{m+\gamma,1})),$$

for any real $\zeta > 1$.

Proposition

Let the height of $\tilde{D}_m^{e,\gamma}(\xi)$ be selected as $L = \frac{e_m}{k}$. Referring to the stochastic process Y_t generated from a point x of the axis of the cone and evolving in $\tilde{D}_m^{e,\gamma}(\xi)$, it holds that

$$P^x\left[|Y_\tau| = \frac{e_m}{k}\right] \geq \frac{j_m(e_m)}{j_m(k|x-\xi|)} \frac{[j_{m+\gamma}(k|x-\xi|)y_{m+\gamma}(k\eta) - j_{m+\gamma}(k\eta)y_{m+\gamma}(k|x-\xi|)]}{[j_{m+\gamma}(e_m)y_{m+\gamma}(k\eta) - j_{m+\gamma}(k\eta)y_{m+\gamma}(e_m)]}$$

Remark

$$P^x\left[|Y_\tau| = \frac{e_m}{k}\right] \geq \frac{j_m(e_m)}{j_m(k|x-\xi|)} \frac{j_{m+\gamma}(k|x-\xi|)}{j_{m+\gamma}(e_m)} \left[1 - \frac{j_{m+\gamma}(k\eta)}{j_{m+\gamma}(k|x-\xi|)} \frac{y_{m+\gamma}(k|x-\xi|)}{y_{m+\gamma}(k\eta)}\right].$$

When the point x is even slightly detached from the inner cup it holds that $\frac{j_{m+\gamma}(k\eta)}{j_{m+\gamma}(k|x-\xi|)} \frac{y_{m+\gamma}(k|x-\xi|)}{y_{m+\gamma}(k\eta)} << 1$. Then the lower bound of the probability P^x_{ext} obtains the form $\frac{j_m(e_m)}{j_m(k|x-\xi|)} \frac{j_{m+\gamma}(k|x-\xi|)}{j_{m+\gamma}(e_m)}$. It is remarkable that this ratio involves the starting point and the height of the exterior cup alone. It is obvious that $\lim_{\gamma\to 0} \frac{j_m(e_m)}{j_m(k|x-\xi|)} \frac{j_{m+\gamma}(k|x-\xi|)}{j_{m+\gamma}(e_m)} = 1$, fact assuring the total accumulation of hitting points on the exterior cup when γ converges to zero.

Stochastic representation of the scattered field (2)

- $h_{m,2}(Y_t) = P_m(\cos(\Theta_t))y_m(k|Y_t|)$
- $$d\widetilde{Y}_t = \widetilde{\mathcal{H}}_m(k|Y_t|)\frac{\widetilde{Y}_t}{|\widetilde{Y}_t|^2}dt - \frac{\sin(\widetilde{\Theta}_t)P_m'(\cos(\widetilde{\Theta}_t))}{|\widetilde{Y}_t|P_m(\cos(\widetilde{\Theta}_t))}\hat{\widetilde{\Theta}}_t dt + dB_t,$$

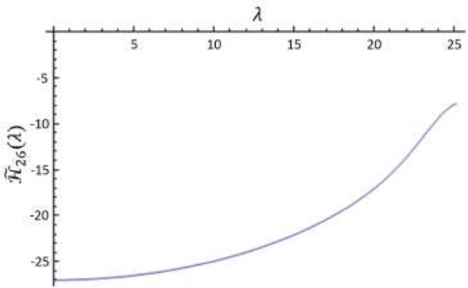

Figure: The increasing negative function $\widetilde{\mathcal{H}}_m(\lambda)$ for $m = 26$.

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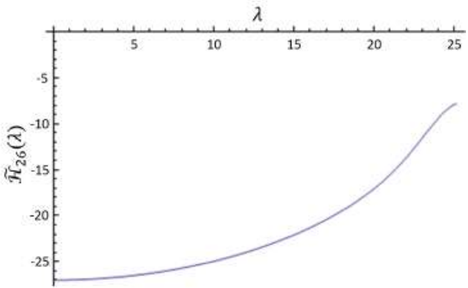

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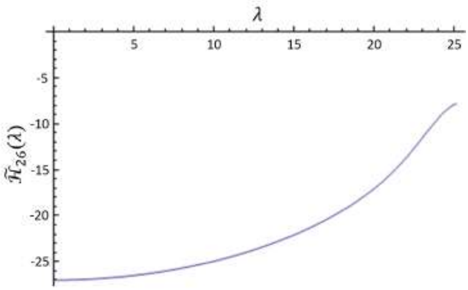
- 

Figure: The increasing negative function $\widetilde{\mathcal{H}}_m(\lambda)$ for $m = 26$.

Stochastic representation of the scattered field (2)

- $$\begin{aligned} u(x) &= \frac{y_m(k|x-\xi|)}{y_m(e_m)} E^x_{\text{ext}}\left[\frac{u(\widetilde{X}_\tau)}{P_m(\cos(\widetilde{\Theta}_\tau))}\right] + \frac{y_m(k|x-\xi|)}{y_m(k\eta)} E^x_{\text{int}}\left[\frac{u(\widetilde{X}_\tau)}{P_m(\cos(\widetilde{\Theta}_\tau))}\right] \\ &+ \frac{y_m(k|x-\xi|)}{\cancel{P_m(\cos(\theta_{m+\gamma,1}))}} E^x_{\text{lat}}\left[\frac{\cancel{u(\widetilde{X}_\tau)}}{y_m(k|\widetilde{Y}_\tau|)}\right],\ x \in \tilde{D}^{e,\gamma}_m(\xi) \end{aligned} \tag{2}$$

- The terms $\frac{y_m(k|x-\xi|)}{y_m(e_m)}$ and $\frac{y_m(k|x-\xi|)}{y_m(k\eta)}$ are unbalanced.
  The same holds for the coefficients $\frac{j_m(k|x-\xi|)}{j_m(e_m)}$ and $\frac{j_m(k|x-\xi|)}{j_m(k\eta)}$.

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Stochastic representation of the scattered field (3): The balanced one.

- A combined auxiliary radial generating function $g_m(\lambda) = C_m j_m(\lambda) - y_m(\lambda)$ leading to $h_{m,3}(\check{Y}_t) = g_m(k|\check{Y}_t|)P_m(\cos\check{\Theta}_t)$
-

$$d\check{Y}_t = \check{\mathcal{H}}_m(k|\check{Y}_t|)\frac{\check{Y}_t}{|\check{Y}_t|^2}dt - \frac{\sin(\check{\Theta}_t)P'_m(\cos(\check{\Theta}_t))}{|\check{Y}_t|P_m(\cos(\check{\Theta}_t))}\hat{\check{\Theta}}_t dt + dB_t, \text{ where}$$

$$\check{\mathcal{H}}_m(\lambda) := \frac{\lambda g'_m(\lambda)}{g_m(\lambda)} = \frac{\lambda[C_m j_{m-1}(\lambda) - y_{m-1}(\lambda)]}{g_m(\lambda)} - (m+1), \quad \lambda > 0.$$

-

$$\begin{aligned} u(x) &= \frac{g_m(k|x-\xi|)}{g_m(e_m)}E^x_{\text{ext}}\left[\frac{u(\check{X}_\tau)}{P_m(\cos(\check{\Theta}_\tau))}\right] + \frac{g_m(k|x-\xi|)}{g_m(k\eta)}E^x_{\text{int}}\left[\frac{u(\check{X}_\tau)}{P_m(\cos(\check{\Theta}_\tau))}\right] \\ &+ \frac{g_m(k|x-\xi|)}{\cancel{P_m(\cos(\theta_{m+\gamma,1}))}}E^x_{\text{lat}}\left[\frac{\cancel{u(\check{X}_\tau)}}{g_m(k|\check{Y}_\tau|)}\right], \; x \in \tilde{D}^{e,\gamma}_m(\xi) \end{aligned} \tag{3}$$

Stochastic representation of the scattered field (3): The balanced one.

- A combined auxiliary radial generating function $g_m(\lambda) = C_m j_m(\lambda) - y_m(\lambda)$ leading to $h_{m,3}(\breve{Y}_t) = g_m(k|\breve{Y}_t|)P_m(\cos\breve{\Theta}_t)$
-

$$d\breve{Y}_t = \breve{\mathcal{H}}_m(k|\breve{Y}_t|)\frac{\breve{Y}_t}{|\breve{Y}_t|^2}dt - \frac{\sin(\breve{\Theta}_t)P_m'(\cos(\breve{\Theta}_t))}{|\breve{Y}_t|P_m(\cos(\breve{\Theta}_t))}\hat{\breve{\Theta}}_t dt + dB_t, \text{ where}$$

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-

$$\begin{aligned} u(x) &= \frac{g_m(k|x-\xi|)}{g_m(e_m)}E^x_{\text{ext}}\left[\frac{u(\breve{X}_\tau)}{P_m(\cos(\breve{\Theta}_\tau))}\right] + \frac{g_m(k|x-\xi|)}{g_m(k\eta)}E^x_{\text{int}}\left[\frac{u(\breve{X}_\tau)}{P_m(\cos(\breve{\Theta}_\tau))}\right] \\ &+ \cancel{\frac{g_m(k|x-\xi|)}{P_m(\cos(\theta_{m+\gamma,1}))}E^x_{\text{lat}}\left[\frac{u(\breve{X}_\tau)}{g_m(k|\breve{Y}_\tau|)}\right]}, \; x \in \tilde{D}^{e,\gamma}_m(\xi) \end{aligned} \qquad (3)$$

Stochastic representation of the scattered field (3): The balanced one.

- A combined auxiliary radial generating function $g_m(\lambda) = C_m j_m(\lambda) - y_m(\lambda)$ leading to $h_{m,3}(\breve{Y}_t) = g_m(k|\breve{Y}_t|) P_m(\cos \breve{\Theta}_t)$
-
$$d\breve{Y}_t = \breve{\mathcal{H}}_m(k|\breve{Y}_t|) \frac{\breve{Y}_t}{|\breve{Y}_t|^2} dt - \frac{\sin(\breve{\Theta}_t) P'_m(\cos(\breve{\Theta}_t))}{|\breve{Y}_t| P_m(\cos(\breve{\Theta}_t))} \hat{\breve{\Theta}}_t dt + dB_t, \text{ where}$$
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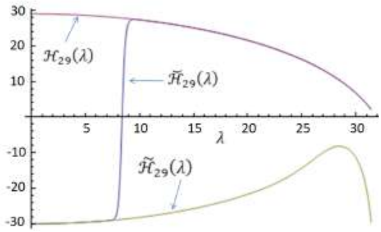
-
$$\begin{aligned} u(x) &= \frac{g_m(k|x-\xi|)}{g_m(e_m)} E^x_{\mathrm{ext}} \left[\frac{u(\breve{X}_\tau)}{P_m(\cos(\breve{\Theta}_\tau))} \right] + \frac{g_m(k|x-\xi|)}{g_m(k\eta)} E^x_{\mathrm{int}} \left[\frac{u(\breve{X}_\tau)}{P_m(\cos(\breve{\Theta}_\tau))} \right] \\ &+ \frac{g_m(k|x-\xi|)}{\cancel{P_m(\cos(\theta_{m+\gamma,1}))}} E^x_{\mathrm{lat}} \left[\frac{\cancel{u(\breve{X}_\tau)}}{g_m(k|\breve{Y}_\tau|)} \right], \ x \in \tilde{D}^{e,\gamma}_m(\xi) \end{aligned} \tag{3}$$

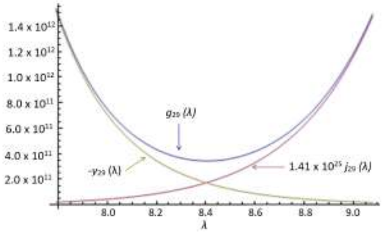
Design of the cone (The direct problem)

- The first concern is the selection of the observation point x and the next step is the determination of the coefficient C_m appearing in g_m by demanding the validity of relation $C_m j_m(k|x-\xi|) = -y_m(k|x-\xi|)$. Then what remains is the determination of the interior and exterior radii η, $\frac{e_m}{k}$. These parameters could be selected arbitrarily but in that case the probabilities of escaping through the shells would be of different order and then the coefficients $\frac{g_m(k|x-\xi|)}{g_m(e_m)}$ and $\frac{g_m(k|x-\xi|)}{g_m(k\eta)}$ of representation (3) would be unbalanced in order to compensate this unfitness. If equipartition of escaping is desired then the one parameter e_m is selected close to the first maximal point of j_m while the other one must be chosen[3] so that $g_m(k\eta) = g_m(e_m)$.

[3] The selection is unique.

The driving terms of the three stochastic representations for $m = 29$. The geometrical parameters have been selected as $k\eta = \pi$ and $e_{29} = 10\pi$. Then, the radial function g_{29} has the form $g_{29}(\lambda) = 1.41 \times 10^{25} j_{29}(\lambda) - y_{29}(\lambda)$.





Equipartition of trajectories

Proposition

It holds that...

$$P^x\left[\left\{|Y_\tau| = \frac{e_m}{k}\right\}\right] \geq \breve{\mathcal{A}}_m(x,\xi,\eta) := \frac{g_m(e_m)}{g_m(k|x-\xi|)} \frac{[j_{m+\gamma}(k|x-\xi|)y_{m+\gamma}(k\eta) - j_{m+\gamma}(k\eta)y_{m+\gamma}(k|x-\xi|)]}{[j_{m+\gamma}(e_m)y_{m+\gamma}(k\eta) - j_{m+\gamma}(k\eta)y_{m+\gamma}(e_m)]}$$

$$\approx \frac{C_m j_m(e_m)}{2C_m j_m(k|x-\xi|)} \frac{j_{m+\gamma}(k|x-\xi|)}{j_{m+\gamma}(e_m)} \underset{\gamma\to 0}{\rightarrow} \frac{1}{2}$$

$$P^x\left[\{|Y_\tau| = \eta\}\right] \geq \breve{\mathcal{B}}_m(x,\xi,\eta) := \frac{g_m(k\eta)}{g_m(k|x-\xi|)} \frac{[j_{m+\gamma}(e_m)y_{m+\gamma}(k|x-\xi|) - j_{m+\gamma}(k|x-\xi|)y_{m+\gamma}(e_m)]}{[j_{m+\gamma}(e_m)y_{m+\gamma}(k\eta) - j_{m+\gamma}(k\eta)y_{m+\gamma}(e_m)]}$$

$$\approx \frac{(-y_m(k\eta))}{2(-y_m(k|x-\xi|))} \frac{y_{m+\gamma}(k|x-\xi|)}{y_{m+\gamma}(k\eta)} \underset{\gamma\to 0}{\rightarrow} \frac{1}{2}$$

Design of the cone (The inverse problem)

The cone $\tilde{D}_m^{e,\gamma}(\xi)$ defines the crucial dimensions η and $L = \frac{e_m}{k}$. Then the coefficient C_m in the definition formula of g_m is chosen such that the function g_m obtains the same values at the end points of its domain, i.e. $g_m(k\eta) = g_m(e_m)$. This means that

$$C_m = \frac{y_m(e_m) - y_m(k\eta)}{j_m(e_m) - j_m(k\eta)}$$

Then the coefficients $\frac{g_m(k|x-\xi|)}{g_m(e_m)}$ and $\frac{g_m(k|x-\xi|)}{g_m(k\eta)}$ of the first two terms of the representation (3) are equal and the equipartition is introduced. What remains is to ascertain that the probabilities of hitting the two cups are comparable. This can be realized if the point x is appropriately selected.

$$C_m j_m(k|x-\xi|) = -y_m(k|x-\xi|) = \frac{1}{2} g_m(k|x-\xi|).$$

The unique point x, is selected to define the initial point x of the stochastic process. Every trajectory emanating from x is subjected, in the beginning of its travel, to a pure Brownian boost since the driving term is locally zero. So its initial directivity obeys to the Brownian law but once it finds an orientation, the driving term obtains abruptly one of the already studied inwards or outwards behavior, pushing the trajectory to cross the corresponding cup.

The discretized Euler scheme

- $$
\begin{aligned}
Y_0 &= (Y_0^{(1)}, Y_0^{(2)}, Y_0^{(3)}) = x \\
Y_{n+1}^{(i)} &= Y_n^{(i)} - \left[\frac{(2m+1)}{|Y_n|^2} Y_n^{(i)} - \frac{k Y_n^{(i)} j_{m-1}(k|Y_n|)}{|Y_n| j_m(k|Y_n|)} \right] \Delta t_n - \frac{P'_{m-1}(\cos\theta_n)}{P_m(\cos\theta_n)} \frac{1}{|Y_n|^2} Y_n^{(i)} \Delta t_n + \Delta B_n^{(i)}, \quad i = 1, 2 \\
Y_{n+1}^{(3)} &= Y_n^{(3)} - \left[\frac{(2m+1)}{|Y_n|^2} Y_n^{(3)} - \frac{k Y_n^{(3)} j_{m-1}(k|Y_n|)}{|Y_n| j_m(k|Y_n|)} \right] \Delta t_n + \frac{m P_{m-1}(\cos\theta_n)}{\cos\theta_n P_m(\cos\theta_n)} \frac{1}{|Y_n|^2} Y_n^{(3)} \Delta t_n + \Delta B_n^{(3)}.
\end{aligned}
$$

- $$
\begin{aligned}
\widetilde{Y}_0 &= (\widetilde{Y}_0^{(1)}, \widetilde{Y}_0^{(2)}, \widetilde{Y}_0^{(3)}) = x \\
\widetilde{Y}_{n+1}^{(i)} &= \widetilde{Y}_n^{(i)} - \left[\frac{(2m+1)}{|\widetilde{Y}_n|^2} \widetilde{Y}_n^{(i)} - \frac{k \widetilde{Y}_n^{(i)} y_{m-1}(k|\widetilde{Y}_n|)}{|\widetilde{Y}_n| y_m(k|\widetilde{Y}_n|)} \right] \Delta t_n - \frac{P'_{m-1}(\cos\theta_n)}{P_m(\cos\theta_n)} \frac{1}{|\widetilde{Y}_n|^2} \widetilde{Y}_n^{(i)} \Delta t_n + \Delta B_n^{(i)}, \quad i = 1, 2 \\
\widetilde{Y}_{n+1}^{(3)} &= \widetilde{Y}_n^{(3)} - \left[\frac{(2m+1)}{|\widetilde{Y}_n|^2} \widetilde{Y}_n^{(3)} - \frac{k \widetilde{Y}_n^{(3)} y_{m-1}(k|\widetilde{Y}_n|)}{|\widetilde{Y}_n| y_m(k|\widetilde{Y}_n|)} \right] \Delta t_n + \frac{m P_{m-1}(\cos\theta_n)}{\cos\theta_n P_m(\cos\theta_n)} \frac{1}{|\widetilde{Y}_n|^2} \widetilde{Y}_n^{(3)} \Delta t_n + \Delta B_n^{(3)}.
\end{aligned}
$$

The discretized Euler scheme

- $$Y_0 = (Y_0^{(1)}, Y_0^{(2)}, Y_0^{(3)}) = x$$
$$Y_{n+1}^{(i)} = Y_n^{(i)} - \left[\frac{(2m+1)}{|Y_n|^2} Y_n^{(i)} - \frac{kY_n^{(i)} j_{m-1}(k|Y_n|)}{|Y_n| j_m(k|Y_n|)} \right] \Delta t_n - \frac{P'_{m-1}(\cos\theta_n)}{P_m(\cos\theta_n)} \frac{1}{|Y_n|^2} Y_n^{(i)} \Delta t_n + \Delta B_n^{(i)}, \quad i = 1, 2$$
$$Y_{n+1}^{(3)} = Y_n^{(3)} - \left[\frac{(2m+1)}{|Y_n|^2} Y_n^{(3)} - \frac{kY_n^{(3)} j_{m-1}(k|Y_n|)}{|Y_n| j_m(k|Y_n|)} \right] \Delta t_n + \frac{mP_{m-1}(\cos\theta_n)}{\cos\theta_n P_m(\cos\theta_n)} \frac{1}{|Y_n|^2} Y_n^{(3)} \Delta t_n + \Delta B_n^{(3)}.$$

- $$\widetilde{Y}_0 = (\widetilde{Y}_0^{(1)}, \widetilde{Y}_0^{(2)}, \widetilde{Y}_0^{(3)}) = x$$
$$\widetilde{Y}_{n+1}^{(i)} = \widetilde{Y}_n^{(i)} - \left[\frac{(2m+1)}{|\widetilde{Y}_n|^2} \widetilde{Y}_n^{(i)} - \frac{k\widetilde{Y}_n^{(i)} y_{m-1}(k|\widetilde{Y}_n|)}{|\widetilde{Y}_n| y_m(k|\widetilde{Y}_n|)} \right] \Delta t_n - \frac{P'_{m-1}(\cos\theta_n)}{P_m(\cos\theta_n)} \frac{1}{|\widetilde{Y}_n|^2} \widetilde{Y}_n^{(i)} \Delta t_n + \Delta B_n^{(i)}, \quad i = 1, 2$$
$$\widetilde{Y}_{n+1}^{(3)} = \widetilde{Y}_n^{(3)} - \left[\frac{(2m+1)}{|\widetilde{Y}_n|^2} \widetilde{Y}_n^{(3)} - \frac{k\widetilde{Y}_n^{(3)} y_{m-1}(k|\widetilde{Y}_n|)}{|\widetilde{Y}_n| y_m(k|\widetilde{Y}_n|)} \right] \Delta t_n + \frac{mP_{m-1}(\cos\theta_n)}{\cos\theta_n P_m(\cos\theta_n)} \frac{1}{|\widetilde{Y}_n|^2} \widetilde{Y}_n^{(3)} \Delta t_n + \Delta B_n^{(3)}.$$

The Inverse Problem: Part I

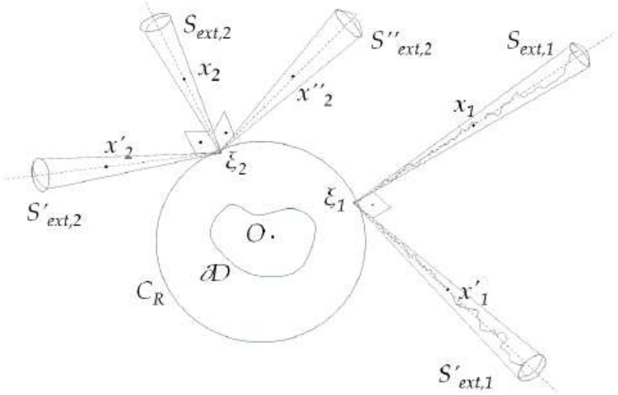


Figure: Transferring data from the far field to the near field region.

The Inverse Problem: Part I

- Instead of applying the stochastic representation to the scattered field u, just do it for the Helmholtz equation vector solution $M(\check{X}_t) = (\check{X}_t - x_1) \times \nabla u(\check{X}_t)$, which is a null field at the starting point x_1.
- $$0 = M(x_1) = \frac{g_m(k|x_1 - \xi_1|)}{g_m(e_m)} E^{x_1}_{\text{ext}}\left[\frac{M(\check{X}_\tau)}{P_m(\cos(\check{\Theta}_\tau))}\right] + \frac{g_m(k|x_1 - \xi_1|)}{g_m(k\eta)} E^{x_1}_{\text{int}}\left[\frac{M(\check{X}_\tau)}{P_m(\cos(\check{\Theta}_\tau))}\right]$$
- $$E^{x}_{\text{int}}\left[\frac{M(\check{X}_\tau)}{P_m(\cos(\check{\Theta}_\tau))}\right] \approx -y_1 \times \nabla u(\xi_1) E^{x}_{\text{int}}\left[\frac{1}{P_m(\cos(\check{\Theta}_\tau))}\right], \quad y_1 = x_1 - \xi_1 .$$

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The Inverse Problem: Part I

$$\hat{y}_1 \times \nabla u(\xi_1) = \frac{1}{|x_1 - \xi_1| E_{\text{int}}^{x_1}\left[\frac{1}{P_m(\cos(\breve{\Theta}_\tau))}\right]} E_{\text{ext}}^{x_1}\left[\frac{(\breve{X}_\tau - x_1) \times \nabla u(\breve{X}_\tau)}{P_m(\cos(\breve{\Theta}_\tau))}\right]$$

$$\hat{y}_1' \times \nabla u(\xi_1) = \frac{1}{|x_1' - \xi_1| E_{\text{int}}^{x_1'}\left[\frac{1}{P_m(\cos(\breve{\Theta}_\tau'))}\right]} E_{\text{ext}}^{x_1'}\left[\frac{(\breve{X}_\tau' - x_1') \times \nabla u(\breve{X}_\tau')}{P_m(\cos(\breve{\Theta}_\tau'))}\right]$$

where $\hat{y}_1'$ is normal to $\hat{y}_1$ and $|x_1' - \xi_1| = |x_1 - \xi_1|$. Determining so $\hat{y}_1 \times \nabla u(\xi_1)$ and $\hat{y}_1' \times \nabla u(\xi_1)$ implies reconstruction of the full vector field $\nabla u(\xi_1)$. The last assertion holds even in the case that the surrounding surface is not spherical and so the vector $\hat{y}_1$ is not necessarily the normal vector on the surface. The same situation can be repeated for every arbitrary point ξ_2 located on C_R. The selection of the secondary perpendicular cone is arbitrary and depends on the availability of data.

The Inverse Problem: Part I

$$\nabla u(\xi) = \frac{1}{|x-\xi| E^x_{\text{int}}\left[\frac{1}{P_m(\cos(\breve{\Theta}_\tau))}\right]} \left[(\hat{y}'\hat{y}'' - \hat{y}''\hat{y}') \cdot E^x_{S_{\text{ext},x}} \left[\frac{(\breve{X}_\tau - x) \times \nabla u(\breve{X}_\tau)}{P_m(\cos(\breve{\Theta}_\tau))} \right] \right.$$

$$\left. - \hat{y}\hat{y}'' \cdot E^{x'}_{S_{\text{ext},x'}} \left[\frac{(\breve{X}'_\tau - x') \times \nabla u(\breve{X}'_\tau)}{P_m(\cos(\breve{\Theta}'_\tau))} \right] \right].$$

where $\hat{y}'' = \hat{y} \times \hat{y}'$.

- In fact, it is not always necessary to use a different set of cones (and far-field subregions) for every particular point ξ of the surrounding near-field surface. This remark can be explained in a converse manner introducing the well known case of the restricted far-field data. The question is to determine the range of influence of this restricted set. Indeed, let us assume that the vector ∇u is measured on a portion S_0 of the remote field regime. As depicted in next Figure , the surface element S_0 defines a truncated cone S_{tr} which divides the surface C_R in two parts, the shadowed one and its complement C_R^+, on which a grid of points ξ_i can be distributed.
- Every such point constitutes the vertex of a cone whose base is the sub-surface S_0. These cones do not have - in their majority - upper spherical cups orientated with their axis $\hat{y}_i$, but this geometrical slight declination does not affect the probabilistic setting developed so far given that the influence of the bases to the evolution of the trajectories applies only to the final steps of their travel. Summarizing, if only $\hat{y}_i \times \nabla u(\xi_i)$ are to be determined (for a plethora of points ξ_i, $i = 1, 2, \ldots, N$ on C_R^+), then the information on S_0 is enough. Moreover, if we are interested in determining the vectors $\nabla u(\xi_i)$ themselves, we evoke the dual perpendicular cones whose bases assembly form a supplementary far field data region of indispensable utility.

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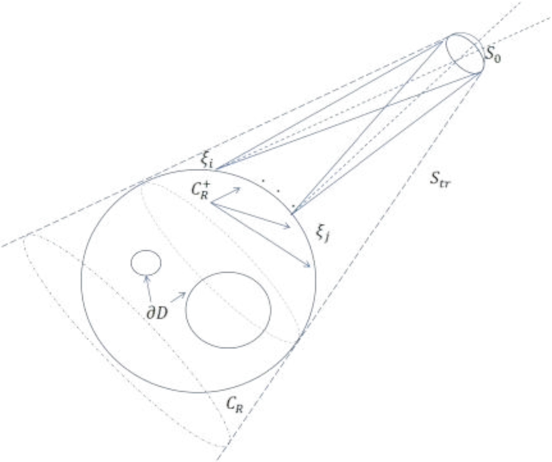

Figure: The region of influence of the data confined on S_0.

Instead of gradients, determine the field itself...

- $$N(x;\alpha) = -\nabla \times M(x)|_{x_j=\alpha} = -\nabla \times ((x-\alpha) \times \nabla u(x))$$

- $$\begin{aligned} v(x;\alpha) &:= (x-\alpha)\cdot N(x;\alpha) \\ &= 2(x-\alpha)\cdot\nabla u(x) + (x-\alpha)(x-\alpha):\nabla\nabla u(x) + k^2|x-\alpha|^2 u(x) \in \ker(\Delta + k^2) \end{aligned}$$

- $$v(\xi_2;\alpha) = -\frac{1}{E^{\alpha}_{\text{int}}\left[\frac{1}{P_m(\cos(\breve{\Theta}^{(\alpha)}_{\tau}))}\right]} E^{\alpha}_{\text{ext}}\left[\frac{v(\breve{X}^{(\alpha)}_{\tau};\alpha)}{P_m(\cos(\breve{\Theta}^{(\alpha)}_{\tau}))}\right], \text{ for } \alpha = x_2, x_2' \text{ and } x_2''$$

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- $$\sum_{\alpha\in\{x,x',x''\}} v(\xi_2;\alpha) = -2|x_2-\xi_2|(\hat{y}_2+\hat{y}_2'+\hat{y}_2'')\cdot\nabla u(\xi_2) + |x_2-\xi_2|^2\Delta u(\xi_2) + 3k^2|x_2-\xi_2|^2 u(\xi_2)$$
$$= -2|x_2-\xi_2|(\hat{y}_2+\hat{y}_2'+\hat{y}_2'')\cdot\nabla u(\xi_2) + 2k^2|x_2-\xi_2|^2 u(\xi_2)$$

- $$u(\xi) = \frac{1}{2k^2|x-\xi|^2 E^x_{\text{int}}\left[\frac{1}{P_m(\cos(\breve{\Theta}^{(x)}_\tau))}\right]}\left\{\sum_{\alpha\in\{x,x',x''\}} E^{\alpha}_{S_{\text{ext},\alpha}}\left[\frac{F_1(\breve{X}^{(\alpha)}_\tau)}{P_m(\cos(\breve{\Theta}^{(\alpha)}_\tau))}\right]\right.$$
$$\left. + \quad (\hat{y}''-\hat{y}')\cdot E^x_{S_{\text{ext},x}}\left[\frac{(\breve{X}^{(x)}_\tau - x)\times\nabla u(\breve{X}^{(x)}_\tau)}{P_m(\cos(\breve{\Theta}^{(x)}_\tau))}\right] - \hat{y}''\cdot E^{x'}_{S_{\text{ext},x'}}\left[\frac{(\breve{X}^{(x')}_\tau - x')\times\nabla u(\breve{X}^{(x')}_\tau)}{P_m(\cos(\breve{\Theta}^{(x')}_\tau))}\right]\right\}$$

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Remark

The scattered field $u(X) = u(X; \hat{k}, k)$, obeys to the Atkinson-Wilcox expansion $u(X) = \frac{e^{ik|X|}}{|X|} \sum_{n=0}^{\infty} \frac{f_n(\hat{X};\hat{k},k)}{|X|^n}$ outside the circumscribing sphere ($|X| > R$), where we encounter the radiation pattern $f_0(\hat{X}; \hat{k}, k) = u_\infty(\hat{X}; \hat{k}, k)$. In second order approximation, we have the following asymptotic form for the remote field:
$u(X) = \frac{e^{ik|X|}}{|X|} \left[f_0(\hat{X}; \hat{k}, k) + \frac{1}{|X|} f_1(\hat{X}; \hat{k}, k) \right] + u_2(X), \ |X|^2 u_2(X) \to 0, \ as \ |X| \to \infty$. The coefficients f_n are related via the well known recursion scheme $2iknf_n = n(n-1)f_{n-1} + \mathbb{B}f_{n-1}$, $n = 1, 2, \ldots$. So the field $F_1(X) = \mathbb{B}u(X)$ is experesed as
$F_1(X) = \mathbb{B}u(X) = \frac{e^{ik|X|}}{|X|} \mathbb{B}f_0(\hat{X}; \hat{k}, k) + O(\frac{1}{|X|^2}) = \frac{e^{ik|X|}}{|X|} 2ikf_1(\hat{X}; \hat{k}, k) + O(\frac{1}{|X|^2})$ as $|X| \to \infty$, involving so the second order approximation f_1 of the remote field expansion. The far field pattern has always the expansion in terms of the spherical harmonics Y_n^m : $f_0 = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} b_n^m Y_n^m$, with coefficients $b_n^m = \int_\Omega f_0(\hat{X}) \overline{Y_n^m}(\hat{X}) d\hat{X}$. ($\Omega$ stands for the unit sphere). Theoretically, in the absence of noise, the coefficients b_n^m are rapidly decaying satisfying the growth condition $\sum_{n=0}^{\infty} \left(\frac{2n}{keR}\right)^{2n} \sum_{m=-n}^{n} |b_n^m|^2 < \infty$. The field $2ikf_1 = \mathbb{B}f_0$ is represented as the expansion $2ikf_1 = -\sum_{n=0}^{\infty} n(n+1) \sum_{m=-n}^{n} b_n^m Y_n^m$. The last expansion represents a stable estimation of f_1 in case that the noise corruption does alter the mentioned before growth behavior to such an extent that the reasonable and much weaker summability condition $\sum_{n=0}^{\infty} n^2(n+1)^2 \sum_{m=-n}^{n} |b_n^m|^2 < \infty$ is not violated.
The same process is followed to construct numerical implementations of the remote vector wave field $(X - x) \times \nabla u(X)$, behaving like $\frac{e^{ik|X|}}{|X|} \hat{X} \times \mathbb{D}f_0(\hat{X})$ as $|X| \to \infty$, where we recognize the spherical surface tangential gradient $\mathbb{D} = \hat{\theta} \frac{\partial}{\partial \theta} + \frac{1}{\sin\theta} \frac{\partial}{\partial \phi}$. Evoking the spherical harmonic expansion of the far field pattern, we find that $(X - x) \times \nabla u(X) \approx \frac{e^{ik|X|}}{|X|} \hat{X} \times \mathbb{D}f_0(\hat{X}) = \frac{e^{ik|X|}}{|X|} \hat{X} \times \sum_{n=0}^{\infty} \sum_{m=-n}^{n} b_n^m \mathbb{D} Y_n^m(\hat{X})$ $= -\frac{e^{ik|X|}}{|X|} \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \sqrt{n(n+1)} b_n^m \mathbb{C}_n^m(\hat{X}), \ |X| \to \infty$, where there emerges, for every pair (n, m), one of the Hansen mutually orthogonal vector spherical harmonics (i.e. the eigenvectors $\mathbb{P}_n^m(\hat{X}) = \hat{X} Y_n^m(\hat{X})$, $\mathbb{B}_n^m(\hat{X}) = \frac{1}{\sqrt{n(n+1)}} \mathbb{D} Y_n^m(\hat{X})$ and $\mathbb{C}_n^m(\hat{X}) = \frac{1}{\sqrt{n(n+1)}} \mathbb{D} Y_n^m(\hat{X}) \times \hat{X}$.)

$$u(\xi) = \frac{1}{2k^2|x-\xi|^2 \sum_{j=1}^{N_{\text{int}}^{(x)}} \left[\frac{1}{P_m(\cos(\breve{\Theta}_j^{(x)}))} \right]} \left\{ \sum_{\alpha \in \{x,x',x''\}} \sum_{i=1}^{N_{\text{ext}}^{(\alpha)}} \left[\frac{F_1(\breve{X}_i^{(\alpha)})}{P_m(\cos(\breve{\Theta}_i^{(\alpha)}))} \right] \right.$$

$$\left. + (\hat{y}'' - \hat{y}') \cdot \sum_{i=1}^{N_{\text{ext}}^{(x)}} \left[\frac{(\breve{X}_i^{(x)} - x) \times \nabla u(\breve{X}_i^{(x)})}{P_m(\cos(\breve{\Theta}_i^{(x)}))} \right] - \hat{y}'' \cdot \sum_{i=1}^{N_{\text{ext}}^{(x')}} \left[\frac{(\breve{X}_i^{(x')} - x') \times \nabla u(\breve{X}_i^{(x')})}{P_m(\cos(\breve{\Theta}_i^{(x')}))} \right] \right\},$$

where $N_{\text{ext}}^{(\alpha)}$ represents the number of exits of the trajectories through the exterior surface of the cone K^{α}. Given that the involved cones are identical then $N_{\text{ext}}^{(x)} = N_{\text{ext}}^{(x')} = N_{\text{ext}}^{(x'')}$. In addition, the theoretically established equipartition of the trajectories, pertaining to their orientation, implies that $N_{\text{ext}}^{(x)} \approx N_{\text{int}}^{(x)} \approx \frac{N}{2}$, fact which is approved by the experiments.

On reconstructing convex scatterers

In the beginning, we consider the primitive case of a spherical scatterer of radius a, centered at the coordinate origin. Being more specific, we consider the simple case of $a = 1$ and $k = 2$ ($\lambda = \pi$ length units). The points ξ_i are sampled uniformly inside a cube $\mathcal{Q}$ centered at the coordinate origin and having edges of length $2\lambda = \frac{4\pi}{k} = 2\pi$. This cube hosts the determinable scatterer.We consider a remote distance $|X|$ of order of $5\lambda = \frac{10\pi}{k} = 5\pi$ where the synthetic data are collected. We focus on the portion of data that is confined on a spherical subregion of the sphere $k|X| = 10\pi$ around the radial direction $\hat{r}_0 = -\hat{k} = \frac{1}{\sqrt{3}}(1, 1, 1)$. The aforementioned choices define the crucial geometric parameter $k|X| = e_m = 10\pi$, defining the range of the remote scattering field and the typical size of the involved cones. So, according to the parametric analysis described in previous sections, the integer m takes the optimal value $m = 29$. The mentioned above range of used data S_0 (on the sphere $k|X| = 10\pi$ and in the vicinity of $\hat{r}_0$) has a spherical polar aperture confined by the angle $\theta_{29,1}$. The vertex of every involved cone K_i is one of the sampling points ξ_i, its interior cup is very close to the vertex via the selection $k\eta = 0.01\pi$. The geometrical feature $k|x_i - \xi_i|$ for the involved cones, is determined to have the exact value 0.2524π. The hitting probabilities are theoretically foreseen as described above to be $P^x\left[\left\{|Y_\tau| = \frac{e_m}{k}\right\}\right] \geq 0.500039$ and $P^x\left[\{|Y_\tau| = \eta\}\right] \geq 0.499774$ and the simulations verified this prediction.

On reconstructing convex scatterers

- Prescribing further the performed numerical experiments, it is noticed that inside the cube $\mathcal{Q}$, a set of $M = 20^3$ uniformly distributed potentially candidate surface points ξ_i has been sampled and for every $i \in \{1, 2, \ldots, M\}$, stochastic experiments have been performed pertaining to the solution of the underlying stochastic differential equations in the cones K_i, K_i' and K_i''. The Monte Carlo realization of the involved expectation terms required at most $N = 10^2$ experiments-repetitions with a typical life-time of traveling inside the cones expressed via the rule $k^2 T = 10^{-2}$. This result is in conjunction with the uniform selection of the parameters $\zeta = 2$ and $\nu = 1$, defining in detail the angle of the cones according to the results presented above. This stochastic implementation leaded to the determination of the stochastic terms $\tilde{u}(\xi_i)$, $i \in \{1, 2, \ldots, M\}$.
- The inversion algorithm consists in constructing and investigating the objective function $G(\xi) = |\tilde{u}(\xi) + e^{ik\hat{k}\cdot\xi}|$. The points ξ_i assigning small values to the functional $G(\xi_i)$ are the supporting points of the surface ∂D. In the following figure, we just plot the level set of the interpolating function $G(\xi)$, representing the set of points satisfying the criterion $G(\xi) = \epsilon$ with $\epsilon \leq 10^{-2}$.

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On reconstructing the sphere

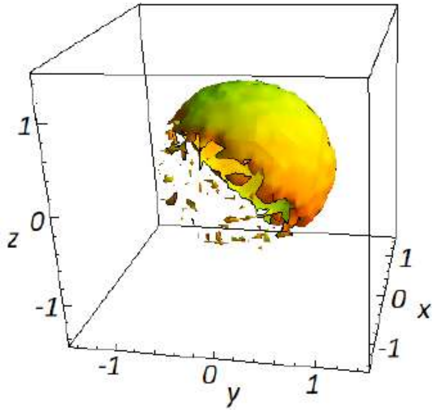

Figure: The reconstruction of the sphere of radius $a = 1$, in the framework of the backscattering case $\hat{r}_0 = -\hat{k} = \frac{1}{\sqrt{3}}(1, 1, 1)$. The principal data are distributed over a surface element of measure $2\pi(5\pi)^2(1 - \cos\theta_{29,1})$ and are supplemented with additional information over the dual cones of type K' and K''.

On reconstructing the ellipsoid $\frac{x^2}{a_1^2} + \frac{y^2}{a_2^2} + \frac{z^2}{a_3^2} = 1$ with considerably unequal semi-axes $a_1 = 4$, $a_2 = 3$, $a_3 = 2$

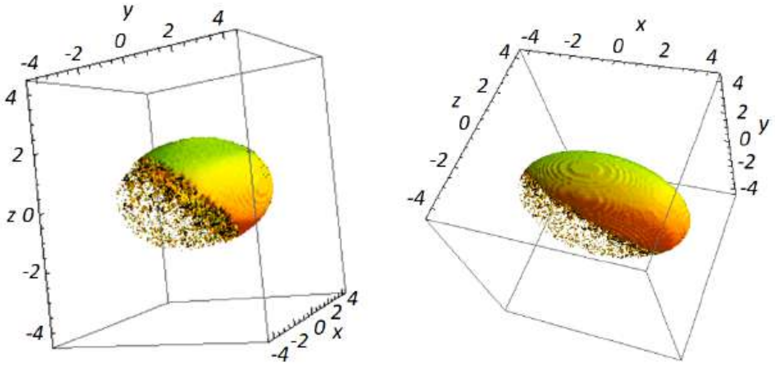

Figure: The reconstruction of the ellipsoidal surface $\frac{x^2}{16} + \frac{y^2}{9} + \frac{z^2}{4} = 1$, for one specific incidence $\hat{k} = -\frac{1}{\sqrt{3}}(1, 1, 1)$ and wave number $k = \frac{1}{5}$.

Part II: An alternative inversion method adequate also for treatment of near field measurements

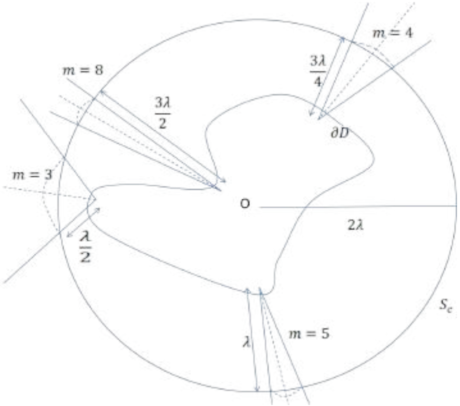

Figure: Every cone has a height that can not exceed $\frac{em}{k} = \frac{em}{2\pi}\lambda$. For several parameters m, the taller versions of the corresponding cones are presented.

$$\hat{y} \times \nabla u(\tilde{\xi};\hat{k}) = \frac{1}{|x-\xi| E^x_{\text{int}}\left[\frac{1}{P_m(\cos(\breve{\Theta}_\tau))}\right]} E^x_{\tilde{S}_m}\left[\frac{(\breve{X}_\tau - x)\times \nabla u(\breve{X}_\tau;\hat{k})}{P_m(\cos(\breve{\Theta}_\tau))}\right] := k\mathcal{M}_m(\tilde{\xi};\hat{k}),$$

with $y = y_\xi = x - \xi$ and $\tilde{\xi} = \xi + \eta\hat{y}$.
The point $\tilde{\xi}$ will be tested as a candidate scatterer's surface point and then the decomposition $\nabla u(\tilde{\xi}) = \hat{n}\frac{\partial u}{\partial n}(\tilde{\xi}) + (\mathbb{I} - \hat{n}\hat{n})\cdot\nabla u(\tilde{\xi})$ will be used, where the normal vector $\hat{n}$ on the scatterer is unknown. In case $\tilde{\xi}$ lies on the surface ∂D, the boundary condition can be evoked, leading to

$$\hat{y} \times \nabla u(\tilde{\xi};\hat{k}) = \hat{y}\times\hat{n}\left(\frac{\partial u}{\partial n}(\tilde{\xi};\hat{k}) + i\hat{n}\cdot(k\hat{k})e^{ik\hat{k}\cdot\tilde{\xi}}\right) - i\hat{y}\times k\hat{k}e^{ik\hat{k}\cdot\tilde{\xi}}$$

Taking the complex conjugate of this equation, we obtain

$$\hat{y} \times \overline{\nabla u\,(\tilde{\xi};\hat{k})} = \hat{y}\times\hat{n}\left(\overline{\frac{\partial u}{\partial n}(\tilde{\xi};\hat{k})} - i\hat{n}\cdot(k\hat{k})e^{-ik\hat{k}\cdot\tilde{\xi}}\right) + i\hat{y}\times k\hat{k}e^{-ik\hat{k}\cdot\tilde{\xi}}$$

Clearly the function $\overline{u}$ is an ingoing wave. We consider now the the outgoing solution $u(x;-\hat{k})$ corresponding to the opposite incidence $(-\hat{k})$. Clearly, it holds that

$$\hat{y} \times \nabla u(\tilde{\xi};-\hat{k}) = \hat{y}\times\hat{n}\left(\frac{\partial u}{\partial n}(\tilde{\xi};-\hat{k}) - i\hat{n}\cdot(k\hat{k})e^{-ik\hat{k}\cdot\tilde{\xi}}\right) + i\hat{y}\times k\hat{k}e^{-ik\hat{k}\cdot\tilde{\xi}}$$

- $$\hat{y} \times \overline{\nabla u(\tilde{\xi};\hat{k})} - \hat{y} \times \nabla u(\tilde{\xi};-\hat{k}) = \hat{y} \times \hat{n} \left(\overline{\frac{\partial u}{\partial n}(\tilde{\xi};\hat{k})} - \frac{\partial u}{\partial n}(\tilde{\xi};-\hat{k}) \right).$$

- 1st case : $\hat{y} \times \overline{\nabla u(\tilde{\xi};\hat{k})} - \hat{y} \times \nabla u(\tilde{\xi};-\hat{k}) \neq 0$
  The direction $\hat{t} = \hat{t}_\xi = \frac{\hat{y} \times \hat{n}}{|\hat{y} \times \hat{n}|}$ can be reconstructed

$$\frac{1}{k}(\hat{y} \times \hat{t}) \cdot \left(\hat{y} \times \nabla u(\tilde{\xi};\hat{k})\right) = \hat{t} \cdot \nabla u(\tilde{\xi};\hat{k}) = -i\hat{t} \cdot \hat{k} e^{ik\hat{k}\cdot\tilde{\xi}}.$$

- 2nd case : $\hat{y} \times \overline{\nabla u(\tilde{\xi};\hat{k})} - \hat{y} \times \nabla u(\tilde{\xi};-\hat{k}) = 0$

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- We define

$$\hat{a}_\xi = \begin{cases} \hat{t}_\xi \text{ if } \overline{\mathcal{M}_m(\tilde{\xi};\hat{k})} \neq \mathcal{M}_m(\tilde{\xi};-\hat{k}) \\ \hat{k} \text{ if } \overline{\mathcal{M}_m(\tilde{\xi};\hat{k})} = \mathcal{M}_m(\tilde{\xi};-\hat{k}) \end{cases}$$

-

$$\mathcal{L}_m(\xi) = \left| \frac{1}{k|x-\xi| E^x_{\text{int}}\left[\frac{1}{P_m(\cos(\breve{\Theta}_\tau))}\right]} E^x_{\tilde{S}_m}\left[\frac{[(\hat{y}_\xi \cdot (\breve{X}_\tau - x))\hat{a}_\xi - (\hat{a}_\xi \cdot (\breve{X}_\tau - x))\hat{y}_\xi] \cdot \nabla u(\breve{X}_\tau;\hat{k})}{P_m(\cos(\breve{\Theta}_\tau))}\right] + i\hat{a}_\xi \cdot \hat{k} e^{ik\hat{k}\cdot\tilde{\xi}} \right|.$$

- The function $\mathcal{L}_m(\xi)$ theoretically vanishes when the point $\tilde{\xi}$, located very close to the vertex ξ, belongs to the surface ∂D. So imposing the constraint $\mathcal{L}(\xi) = \epsilon$, with $\epsilon << 1$ and sampling over a grid of candidate surface points ξ_i, $i = 1, 2, \ldots, M$ leads to the construction of level sets describing the surface of the scatterer.

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Reconstruction of the cubic surface

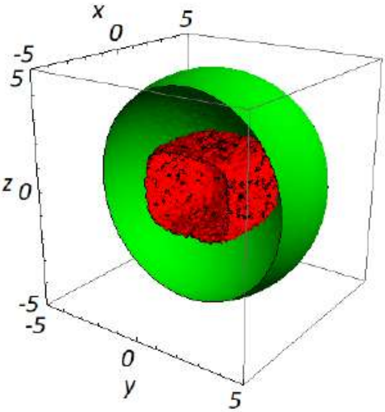

Figure: The reconstruction of the surface of the cubic scatterer with semi-edges $2, \sqrt{3}, \sqrt{2}$ length-units in cartesian coordinates. To reveal transparently the structure of the inversion, the region $x > -1.8$ has been illustrated.

Reconstruction of a disconnected scatterer

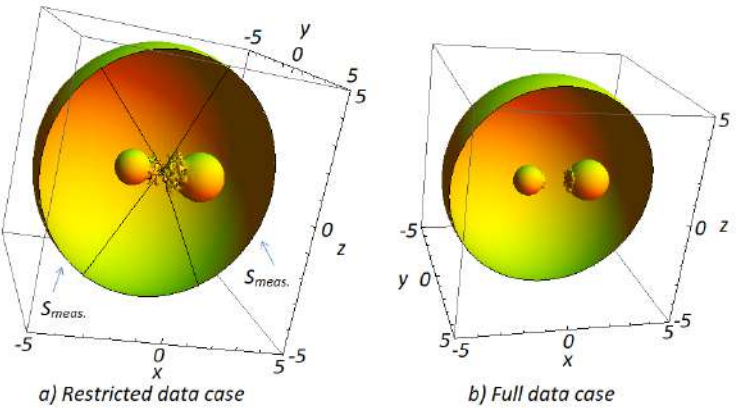

Figure: Reconstruction for the disconnected scatterer case: Detecting two spheres with radii equal to 1 and $\frac{1}{\sqrt{2}}$ centered at the points $x = 2$ and $x = -1$ of the $x-$axis, respectively.

Conclusion: On comparing the two inversion schemes

The alternative algorithm could be used as well in the realm of far field measurements in the case of convex star-shaped scatterers. Indeed, the alternative to detour the implication of additional measurement sets on the dual conical surfaces of the first approach emerges via the current implementation. Comparing the approaches, their particular ingredients are revealed. The dual cones approach is based on one specific excitation $\hat{k}$ and three perpendicular cones providing data on three separate conical spherical cups in the far-field regime. In addition, special treatment of the data is required to construct the field $F_1(\check{X}_{\tau}^{(\alpha)})$ appeared in the first approach. On contrary, the methodology of this section involves measurements generated by two opposite incidence directions $\hat{k}$ and $(-\hat{k})$ every time but restricted on the exterior cup of a single cone for every particular group of Monte Carlo stochastic experiments. Moreover, no need to interfere with the spherical harmonic expansion of the acoustic field is necessary any more in order to construct the auxiliary field F_1. Phenomenically, the second method seems privileged but a more attentive examination reveals some intrinsic special properties. In every realization of the second approach, a quantitative criterion has to be examined defining every time the choice of $\hat{a}_{\xi}$. Although this repeated "if" structure of the second algorithm does not impose some essential numerical burden, the first approach is totally free of any intrinsic geometric constraint.

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- Antonios Charalambopoulos; Leonidas Gergidis; Eleftheria Vassilopoulou : A Conditioned Probabilistic Method for the Solution of the Inverse Acoustic Scattering Problem, Mathematics 2022, 10, 1383.

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Thank you very much.