Uniqueness for recovering the refractive index from far field data or from the knowledge of transmission eigenvalues

Drossos Gintides

Department of Mathematics
National Technical University of Athens
Greece
MATH @ NTUA Summer School 2023, June 2023
Mathematical Theory of Inverse Problems and Applications

Inverse problems

We will discuss two ways of approaching the uniqueness issue of the inverse problem of determining the refractive index $n(x)$ of an inhomogeneity from scattering related information.

- Unique determination of $n(x)$ from far field data for many or a few incident plane waves.
- Determination of $n(x)$ based on the complete or partial knowledge of transmission eigenvalues.

Basic features, practical interest of uniqueness theorems and special techniques and possible connections between the two approaches

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Outline

Part 1: Uniqueness in inverse scattering

1. Theorem for full far field data
2. Karp's theorem for inhomogeneous domains
3. Unique determination of a dielectric disk

Part 2: Inverse spectral problems

1. Inverse Sturm-Liouville eigenvalue problem
2. Uniqueness theorems
3. Inverse transmission eigenvalue problem
4. Uniqueness theorems

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Part 1: Uniqueness of Inverse Scattering Problem

Inverse problem: Assume that we know all far field pattern $u_\infty(\hat{x}, d)$ for all $\hat{x}, d \in \mathbb{S}^2$ and a fixed wave number k. Is this information enough to uniquely determine the refractive index $n(x)$ of a scattering process?

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The answer is positive. In $\mathbb{R}^3$: Nachman (1988), Novikov (1988), Ramm (1988).
In $\mathbb{R}^2$: Bukhgeim (2008).
We will discuss the problem in $\mathbb{R}^3$ and a version due to Hähner (1996).

Part 1: Uniqueness of Inverse Scattering Problem

Some necessary tools: A completeness property of products of entire harmonic functions:

Theorem (Calderón)

If h_1 and h_2 are entire harmonic functions, then the set $h_1 h_2$ is complete in $L^2(D)$ for any bounded domain $D \subset \mathbb{R}^3$.

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Proof: a nice application of Fourier Integral Theorem!
For the uniqueness theorem, we need a similar property, but instead for products $v_1 v_2$ of solutions to equations $\Delta v_1 + k^2 n_1 v_1 = 0$ and $\Delta v_2 + k^2 n_2 v_2 = 0$.

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Uniqueness of Inverse Scattering Problem

For the cube $Q := [-\pi, \pi]^3 \subset \mathbb{R}^3$, the functions

$$e_a(x) := \frac{1}{\sqrt{2\pi}^2} e^{ia \cdot x}, \quad a \in \tilde{Z}^3$$

provide an orthonormal basis for $L^2(Q)$, where

$$\tilde{\mathbb{Z}}^3 := \{a = b - (0, \frac{1}{2}, 0) : \; b \in \mathbb{Z}^3\}$$

If $f \in L^2(Q)$, then we denote its Fourier coefficients with respect to e_a by $\hat{f}_a$.

Uniqueness of Inverse Scattering Problem

Using these definitions, it can be shown that

Theorem

Let $t > 0$ and $\zeta = t(1, i, 0) \in \mathbb{C}^3$. Then,

$$G_\zeta f = -\sum_{a \in \tilde{\mathbb{Z}}^3} \frac{\hat{f}_a}{a \cdot a + 2\zeta \cdot a} e_a$$

defines an operator $G_\zeta : L^2(Q) \to H^2(Q)$ such that $\|G_\zeta f\|_{L^2(Q)} \leq \frac{1}{t}\|f\|_{L^2(Q)}$ and $\Delta G_\zeta f + 2i\zeta \cdot \nabla G_\zeta f = f$ in the weak sense for all $f \in L^2(Q)$.

Uniqueness of Inverse Scattering Problem

The previous theorem is useful in proving the following result

Lemma

Let D be an open ball centered at the origin such that $supp(1-n) \subset D$. Then there exists $C > 0$ such that for each $z \in \mathbb{C}^3$ that satisfies $z \cdot z = 0$ and $|Re\ z| \geq 2k^2 \|n\|_\infty$ there exists a solution $v \in H^2(D)$ to the equation

$$\Delta v + k^2 nv = 0, \quad in\ D$$

of the form $v(x) = e^{iz\cdot x}[1 + w(x)]$, where $\|w\|_{L^2(D)} \leq \frac{C}{|Re\ z|}$

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Uniqueness of Inverse Scattering Problem

Returning to the scattering problem, we have the following completeness result

Lemma

Let B, D are two balls with center at the origin, such that they contain supp$(1-n)$ and $\overline{B} \subset D$. Then the family of total fields $\{u(.,d)\ d \in \mathbb{S}^2\}$ that solve the scattering problem for an incident plane wave $e^{ikx\cdot d}$ is complete in the closure of the set

$$H := \{v \in H^2(D) : \Delta v + k^2 n v = 0,\ D\}$$

with respect to the $L^2(B)$-norm.

Uniqueness of Inverse Scattering Problem

This completeness result is essential in proving the following uniqueness theorem in $\mathbb{R}^3$, which requires infinitely many incident plane waves.

Theorem (A. Nachmann, R. Novikov and A. G. Ramm)

The refractive index $n(x)$ is uniquely determined by a knowledge of the far field pattern $u_\infty(\hat{x}, d)$ for all $\hat{x}, d \in \mathbb{S}^2$ and a fixed wave number k.

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Proof for uniqueness theorem

Assume n_1, n_2 are two refractive indices such that $u_{1,\infty}(.,d) = u_{2,\infty}(.,d)$ for $d \in \mathbb{S}^2$. If $B \subset D$ are two open balls that have center at origin and contain the support of n_1, n_2, from Rellich's Lemma we have that

$$u_1(.,d) = u_2(.,d), \text{ in } \mathbb{R}^3 \setminus B$$

and for all directions $d \in \mathbb{S}^2$. Hence, if we define $u := u_1 - u_2$ it satisfies the boundary conditions $u = \frac{\partial u}{\partial \nu} = 0$ on ∂B and the equation

$$\Delta u + k^2 n_1 u = k^2 (n_2 - n_1) u_2 \text{ on } B$$

Proof for uniqueness theorem

If we combine the latter with the differential equation for $\tilde{u}_1 := u_1(., \tilde{d})$ we obtain

$$k^2 \tilde{u}_1 u_2 (n_2 - n_1) = \tilde{u}_1 (\Delta u + k^2 n_1 u) = \tilde{u}_1 \Delta u - u \Delta \tilde{u}_1$$

From Green's theorem and boundary values, we have that

$$\int_B u_1(., \tilde{d}) u_2(., d)(n_1 - n_2) dx = 0$$

Proof for uniqueness theorem

From the previous Lemma, this implies that

$$\int_B v_1 v_2 (n_1 - n_2) dx = 0$$

for all $H^2(D)$ solutions of the equations $\Delta v_1 + k^2 n_1 v_1 = 0$, $\Delta v_2 + k^2 n_2 v_2 = 0$ in D. For a given $y \in \mathbb{R}^3 \setminus \{0\}$ and $\rho > 0$, we select vectors $a, b \in \mathbb{R}^3$ such that $\{y, a, b\}$ is an orthogonal basis in $\mathbb{R}^3$ such that $|a| = 1$ and $|b|^2 = |y|^2 + \rho^2$. Then, if we define

$$z_1 := y + \rho a + ib, \quad z_2 := y - \rho a - ib$$

Proof for uniqueness theorem

We calculate

$$z_j \cdot z_j = |Rez_j|^2 - |Imz_j|^2 + 2iRez_j \cdot Imz_j = |y|^2 + \rho^2 - |b|^2 = 0$$

and

$$|Rez_j|^2 = |y|^2 + \rho^2 \geq \rho^2$$

Now, we use the solutions v_1, v_2 corresponding to the refractive indices n_1, n_2 and the vectors z_1, z_2 that are described by a previous lemma. By substituting into the last integral and since $z_1 + z_2 = 2y$, we have that

$$\int_B e^{2iy\cdot x}[1 + w_1(x)][1 + w_2(x)][n_1(x) - n_2(x)]dx = 0$$

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Proof for uniqueness theorem

Sending $\rho \to \infty$, by using the inequality

$$\|w_j\|_{L^2(D)} \leq \frac{C}{|Rez_j|}$$

and $|Rez_j| \geq \rho$, we have

$$\int_B e^{2iy\cdot x}[n_1(x) - n_2(x)]dx = 0$$

for all $y \in \mathbb{R}^3$. From the Fourier integral theorem, we conclude that $n_1 = n_2$ in B.

Uniqueness for special geometries

Under appropriate symmetry assumptions for the far field patterns, the corresponding scatterer must be spherical.
First we state a version of this result for the Dirichlet problem and afterwards its extension to the Neumann and inhomogeneous medium problems.

Theorem (Karp's Theorem for the Dirichlet problem)

Suppose that $D \subset \mathbb{R}^2$ is sound soft and the far field pattern is of the form

$$F(k;\theta,a) = F_0(k;\theta - a)$$

for some function F_0. Then, D is a disk.

Uniqueness for special geometries

Theorem (D. Colton and A. Kirsch (1988))

Let the scatterer D be sound-hard and suppose that $F(k;\theta,a) = F_0(k;\theta - a)$ holds for some fixed wavenumber k and all $a \in [-\pi,\pi]$, $\theta \in [-\pi,\pi]$. Then, D is a disk.

Uniqueness for special geometries

For the case of an inhomogeneous medium we also have the following result

Theorem (D. Colton and A. Kirsch (1988))

Suppose that F is the far field pattern corresponding to an inhomogeneous medium with continuously differentiable refractive index $n(x)$ and $F(k;\theta,a) = F_0(k;\theta - a)$ is satisfied for all $k > 0$ and all $a \in [-\pi,\pi]$, $\theta \in [-\pi,\pi]$. Then, $m(x) := 1 - n(x)$ is spherically stratified, that is $m(x) = m_0(r)$ for some function m_0.

Unique determination of a dielectric disk

Consider the transmission problem of finding $u \in H^1_{loc}(\mathbb{R}^2 \backslash \overline{D})$ and $v \in H^1(D)$ that solve

$$\Delta u + k_0^2 u = 0, \quad \text{in } \mathbb{R}^2 \backslash \overline{D}, \quad \Delta v + k_d^2 v = 0, \quad \text{in } D$$

such that

$$u = v, \quad \frac{\partial u}{\partial \eta} = \frac{\partial v}{\partial \eta} \text{ on } \partial D$$

where $u = u^i + u^s$ and u^s satisfies the Sommerfeld radiation condition and $k_0, k_d > 0$, $k_0 \neq k_d$. The inverse (obstacle) problem is given the far field u_∞ for only one incident plane wave with incident direction $d \in \mathbb{S}^1$, to determine the boundary ∂D of the dielectric scatterer D.

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Unique determination of a dielectric disk

Theorem (Kress and Altundag, 2012)

A dielectric disk is uniquely determined by the far field pattern for one incident plane wave.

Unique determination of a dielectric disk

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Proof.

Using polar coordinates, the Jacobi - Anger expansion provides a nice expansion for the incident plane wave:

$$e^{ik_0 x\cdot d} = \sum_{n=-\infty}^{\infty} i^n J_n(k_0\rho)e^{in\theta}, x \in \mathbb{R}^2$$

where the J_n denote the Bessel functions of order n. □

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Proof.

For a disk of radius R centered at the origin we have

$$u^s = \sum_{n=-\infty}^{\infty} i^n \frac{k_0 J_n(k_d R) J_n'(k_0 R) - k_d J_n(k_0 R) J_n'(k_0 R)}{k_d H_n^{(1)}(k_0 R) J_n'(k_d R) - k_0 J_n(k_d R) (H_n^{(1)})'(k_0 R)} H_n^{(1)}(k_0 \rho) e^{in\theta}$$

for $|x| \geq R$. $\square$

Unique determination of a dielectric disk

Also,

$$v = \frac{2}{\pi R} \sum_{n=-\infty}^{\infty} \frac{i^{n-1}}{k_d H_n^{(1)}(k_0 R) J_n'(k_d R) - k_0 J_n(k_d R)(H_n^{(1)})'(k_0 R)} J_n^{(1)}(k_d \rho) e^{in\theta}$$

for $|x| \leq R$.

- It can be seen that the scattered wave has an extension into the interior of D with an exception at the origin.
- If the far field for one incident wave coincides for two disks D_1 and D_2 with different centers, then $u^s \equiv 0$.

Unique determination of a dielectric disk

- By relating the direct scattering problem to solutions of the interior transmission eigenvalue problem for $D = D_1$ corresponding to a piecewise refractive index $n = 1$ in $\mathbb{R}^2 \backslash \overline{D}$ and $n = k_d \backslash k_0$ in D, it can be shown that the two disks must have the same center. The pairs in the expansions for u^1 and v can be considered as solutions of the transmission eig. problem and are linearly independent. For a real-valued refractive index interior transmission eigenvalues have finite multiplicity. Contradiction and therefore the two disks must have the same center.
- Finally, to show that D_1 and D_2 have the same radius, we observe that by symmetry the far field pattern from scattering of a plane wave only depends on the angle between the incident and observation directions. As a consequence, knowledge of the far field for one incident direction implies knowledge for all incident directions. The theorem follows from the uniqueness result for all incident directions.

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Part 2: Inverse Spectral Problems

- Inverse Spectral Problem = Determine certain properties of a system, from a set of spectral data (eigenvalues - eigenfunctions).
  Inverse spectral problems are not well posed.
- Typical ISP in scattering theory:
  Determine the refractive index from a set of eigenvalues.
- Inverse Sturm-Liouville problems:
  Continuous refractive index: Rundell, Sacks, Hald ....
  Discontinuous refractive index: Hald, Kobayashi, Willis, Shahriari, Akbarfam and Teschl, ...

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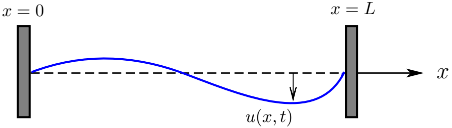
Physical motivation: Sturm-Liouville problem

Inverse Spectral Problems:
recover geometrical or physical/material properties from spectral data

Example 1:

Eigenvalue problem:



$$v''(x) + \lambda\rho(x)v(x) = 0, \quad 0 < x < L$$
$$v'(0) - hv(0) = v'(L) + Hv(L) = 0.$$

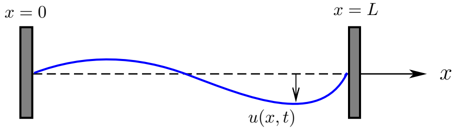
Direct problem: Determine the infinite number of eigenvalues $\{\lambda_i\}_{i=1}^{\infty}$ and corresponding eigenfunctions $\{v_i(x)\}_{i=1}^{\infty}$ = spectral data.

Inverse problem: Determine the density function $\rho(x)$ from spectral data.

Physical motivation: Sturm-Liouville problem

Inverse Spectral Problems:
recover geometrical or physical/material properties from spectral data

Example 1:
Using Liouville's transform the previous problem can be defined as a Sturm-Liouville eigenvalue problem:



$$-u''(x) + q(x)u(x) = \lambda u(x), \quad 0 < x < L$$
$$u'(0) - hu(0) = u'(L) + Hu(L) = 0.$$

Direct problem: Determine the infinite number of eigenvalues $\{\lambda_i\}_{i=1}^{\infty}$ and corresponding eigenfunctions $\{u_i(x)\}_{i=1}^{\infty}$ = spectral data.
Inverse problem: Determine the potential $q(x)$ from spectral data.

Liouville transform

Auxilliary initial value problem Let $v(r) = v(r;\lambda)$ be the unique solution of the initial-value problem

$$v''(r) + \lambda n(r) v(r) = 0,$$

$$v(0) = 0, \qquad v'(0) = 1.$$

Liouville transformation

$$\zeta := \int_0^r \sqrt{n(\eta)}\, d\eta, \qquad z(\zeta) = n(r)^{1/4} v(r),$$

transforms the initial-value problem to:

$$z''(\zeta) - p(\zeta) z(\zeta) + \lambda z(\zeta) = 0,$$

$$z(0) = 0, \qquad z'(0) = \frac{1}{n(0)^{1/4}},$$

where $p(\zeta) = \frac{1}{4}\frac{n''(r)}{n(r)^2} - \frac{5}{16}\frac{n'(r)^2}{n(r)^3}$

Uniqueness:

Determine $q(x)$ from spectral data $\{\lambda_i\}_{i=1}^{\infty}, \{\mu\}_{i=1}^{\infty}$ where λ_i correspond to H and μ_i to a constant H' where $H' \neq H$.

Uniqueness G. Borg, 1945, improvements I. M. Gelfand and B. M. Levitan, 1951, K. Chadan, D. Colton, L. Paivarinta, W. Rundell, 1997......

Applications:

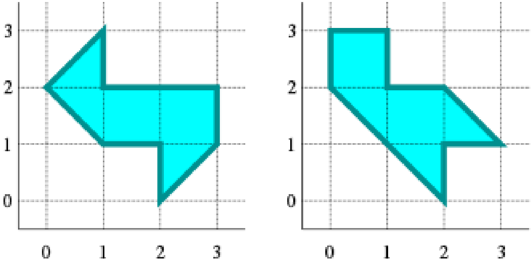
Microphones and sound systems, quantum theory, heliosseismology, solutions of KdV equation, ...

Example 2: $\mathbb{R}^2$

Famous question by Mark Kac : "Can One Hear the Shape of a Drum?" (1966)

For Dirichlet problem.

Answer negative. C. Gordon, D. Webb, and S. Wolpert (1992)



Example 3: $\mathbb{R}^3$

Check by sound experiment if a watermelon is ripe...complicated because depends on the shape and the density but farmers know...

Part 2: Inverse Spectral Problems

1. Inverse spectral problem (ISP) for transmission eigenvalues: Determine $n(x)$ from transmission eigenvalues from a complete or partial knowledge of transmission eugenvalues

Transmission Eigenvalue Problem

Find (w, v) such that

$$\begin{aligned}
\Delta w + k^2 n(x) w = 0 \quad & \text{in} \quad D \\
\Delta v + k^2 v = 0 \quad & \text{in} \quad D \\
w = v \quad & \text{on} \quad \partial D \\
\frac{\partial w}{\partial \nu} = \frac{\partial v}{\partial \nu} \quad & \text{on} \quad \partial D
\end{aligned}$$

It is a nonstandard eigenvalue problem

- If $n = 1$ the interior transmission problem is degenerate
- If $\Im(n) > 0$ in $\overline{D}$, there are no real transmission eigenvalues.

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Spherically Symmetric Medium

We consider the interior eigenvalue problem for a ball of radius a with index of refraction $n(r)$

$$\begin{aligned} \Delta w + k^2 n(r) w &= 0 && \text{in } B \\ \Delta v + k^2 v &= 0 && \text{in } B \\ w &= v && \text{on } \partial B \\ \frac{\partial w}{\partial r} &= \frac{\partial v}{\partial r} && \text{on } \partial B \end{aligned}$$

where $B := \{ x \in \mathbb{R}^3 : \; |x| < a \}$.

Spherically Symmetric Medium

Separation of variables:

$$v_l(r,\theta) = a_\ell j_\ell(kr) P_\ell(\cos\theta) \qquad \text{and} \qquad wl(r,\theta) = a_\ell Y_\ell(kr) P_\ell(\cos\theta)$$

j_ℓ is a spherical Bessel function and Y_ℓ is the solution of

$$Y_\ell'' + \frac{2}{r} Y_\ell' + \left(k^2 n(r) - \frac{\ell(\ell+1)}{r^2} \right) Y_\ell = 0$$

such that $\lim_{r \to 0} \left(Y_\ell(r) - j_\ell(kr) \right) = 0$.

Spherically Symmetric Medium

To determine the transmission eigenvalues we need to find an appropriate non trivial pair of functions $v_l(r,\theta), w_l(r,,\theta)$, satisfying the transmission eigenvalue boundary conditions and due to linearity the wave number must be such that the determinant :

$$d_l(k) := det \begin{pmatrix} Y_\ell(a) & -j_\ell(ka) \\ Y'_\ell(a) & -kj'_\ell(ka) \end{pmatrix} = 0$$

Any determinant $d_l(k), l = 0, ..., \infty$ is a generator for a specific subset of transmission eigenvalues.

Spherically Symmetric Medium

Values of k such that $d_\ell(k)$ has the asymptotic behavior

$$d_\ell(k) = \frac{1}{a^2 k\,[n(0)]^{\ell/2+1/4}} \sin\left(ka - k\int_0^a [n(r)]^{1/2} dr\right) + O\left(\frac{\ln k}{k^2}\right)$$

as $k \to \infty$

Asymptotic relations for real tr. eigenvalues - continuous n

First Results:

Let $A := \int_0^a \sqrt{n(r)}dr$, a the radius, $n(r) > 0, n \in C^1[0,b], n'' \in L^2[0,b]$.
From McLaughlin-Polyakov (1994):

$$k_j^2 = \frac{j^2\pi^2}{(A-a)^2} + O(1), \quad for\ p \in L^2(0,A)$$

(For self-adjoint problems Hald - McLaughlin (1989)).
$\Rightarrow$ if two transmission problems for $n_1(r)$ and $n_2(r)$ have the same infinite set of transmission eigenvalues then $A_1 = A_2$.

Inverse Spectral Problem - continuous n

Theorem (Cakoni - Colton - Gintides, (2010))

If $n(0)$ is given then $n(r)$ is uniquely determined from the knowledge of all transmission eigenvalues, $n \in C^2[0, \infty)$, radius a.

Proof:

Integral representation

Y_l can be written in the form:

$$Y_l(r) = j_l(kr) + \int_0^r G(r, s, k) j_l(ks) ds$$

where G satisfies the following problem:

The Uniqueness Theorem for the Inverse Problem

Goursat Problem

$$r^2\left[\frac{\partial^2 G}{\partial r^2}+\frac{2}{r}\frac{\partial G}{\partial r}+k^2 n(r) G\right]=s^2\left[\frac{\partial^2 G}{\partial s^2}+\frac{2}{s}\frac{\partial G}{\partial s}+k^2 G\right],\quad 0<s\leq r<1$$

$$G(r,r,k)=\frac{k^2}{2r}\int_0^r tm(t)dt,\quad G(r,s,k)=O\left((rs)^{1/2}\right)$$

- $G(r,s;k)$ continuous in $0\leq s\leq r<1$
- $G(r,s;k)$ is entire function of k of exponential type

$$G(r,s,k)=\frac{k^2}{2\sqrt{rs}}\int_0^{\sqrt{rs}} tm(t)dt\left(1+O(k^2)\right)$$

where $m(r)=1-n(r)$

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The Uniqueness Theorem for the Inverse Problem

$$j_\ell(kr) = \frac{\sqrt{\pi}(kr)^\ell}{2^{\ell+1}\Gamma(\ell+3/2)}\left(1 + O(k^2r^2)\right)$$

shows that

$$c_{2\ell+2}\left[\frac{2^{\ell+1}\Gamma(\ell+3/2)}{\sqrt{\pi}a^{(\ell-1)/2}}\right]^2 =$$

$$a\int_0^a \frac{d}{dr}\left(\frac{1}{2\sqrt{rs}}\int_0^{\sqrt{rs}} \rho\, m(\rho)\, d\rho\right)_{r=a} s^\ell\, ds$$

$$-$$

$$\ell\int_0^a \frac{1}{2\sqrt{as}}\int_0^{\sqrt{as}} \rho\, m(\rho)\, d\rho\, s^\ell\, ds + \frac{a^\ell}{2}\int_0^a \rho\, m(\rho)\, d\rho.$$

The Uniqueness Theorem for the Inverse Problem

After tedious calculation involving a change of variables and interchange of orders of integration, simplifies to

$$c_{2\ell+2} = \frac{\pi}{2^{\ell+1} a^2 \Gamma(\ell + 3/2)} \int_0^a \rho^{2\ell+2}\, m(\rho)\, d\rho.$$

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The Uniqueness Theorem for the Inverse Problem

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$$c_{2\ell+2} = \frac{\pi}{2^{\ell+1} a^2 \Gamma(\ell + 3/2)} \int_0^a \rho^{2\ell+2}\, m(\rho)\, d\rho.$$

That means the coefficients of the first term in the power expansion contains information about the contrast = refractive index.
But how to relate with the transmission eigenvalues?

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The Uniqueness Theorem for the Inverse Problem

The answer is given by Hadamard:

The Uniqueness Theorem for the Inverse Problem

The answer is given by Hadamard:

Hadamard's Factorization Theorem

Let $f(z)$ be an entire function of z with order $\rho \geq 0$
that is for all $z : |z| \geq r$ satisfy $|f(z)| \leq ce^{|z|^{\rho}}$. Then $f(z)$ can be represented as an infinite product

$$f(z) = z^{l} e^{P_l(z)} \prod_{n=0}^{\infty} \left(1 - \frac{z}{z_n}\right) e^{\sum_{m=1}^{p} \frac{z^m / z_n^m}{m}}, \;\; z_n \text{ are the roots of the function.}$$

where l is the multiplicity of zero as a root of $f(z)$, $P_l(z)$ is a polynomial of degree less than or equal to ρ .

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The Uniqueness Theorem for the Inverse Problem

Example:

$$sin(\pi z) = \pi z \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2}\right)$$

because $sin(\pi z)$ has 0 with multiplicity one, roots $\pm n$.

The Uniqueness Theorem for the Inverse Problem

- $d_l(k)$ is an entire function of k with order 1

Hadamard's Factorization Theorem

$$d_l(k) = k^{2l+2} c_{2l+2} \prod_{n=1}^{\infty} \left(1 - \frac{k^2}{k_{nl}^2}\right), \quad k_{nl} \text{ are the trans. eigen.}$$

- The asymptotic form of G, y_l and j_l imply the identity:

$$c_{2l+2} = \frac{\pi}{(2^{l+1} a^2 \Gamma(l + 3/2))^2} \int_0^a t^{2l+2} m(t) dt.$$

All k_{nl} are known, so, $d_l(k)/c_{2l+2}$ is known

$$\frac{d_l(k)}{c_{2l+2}} = \frac{\gamma_l}{a^2 k} \sin(k - kA) + O\left(\frac{\ln k}{k^2}\right)$$

where $\gamma_l := 1/(c_{2l+2} n(0)^{l/2+1/4})$
Since $d_l(k)/c_{2l+2}$ is known γ_l is uniquely determined $\forall\ l \in \mathbb{N}$

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The Uniqueness Theorem for the Inverse Problem

▷ Finally: we have the representation

$$\int_0^a t^{2l+2} m(t) dt = \frac{(2^{l+1}\Gamma(l+3/2))^2}{n(0)^{l/2+1/4}\gamma_l \pi}$$

From Müntz's theorem $m(t)$ is uniquely determined.

The Uniqueness Theorem for the Inverse Problem

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From Müntz's theorem $m(t)$ is uniquely determined.
A good reference is the book of P.J. Davis, Interpolation and Approximation, Dover, New York, 1975.

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Spherically symmetric and discontinuous $n(r)$

▷ Based on results of Gintides - Pallikarakis[1] and Pallikarakis[2]

[1] The inverse transmission eigenvalue problem for a discontinuous refractive index, Inverse Problems, 2017.

[2] The inverse spectral problem for the reconstruction of the refractive index from the interior transmission problem, Ph.D. thesis, NTUA, 2017.

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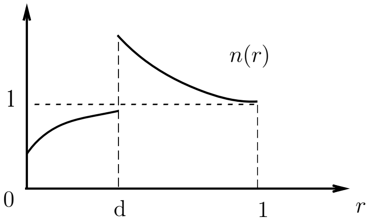
Discontinuous refractive index:

$$n(r) \text{ is } C^2 \text{ in } [0, d) \text{ and } (d, 1],\ n(r) = 1 \text{ for } r \geq 1,\ n'(1) = 0$$



[1] The inverse transmission eigenvalue problem for a discontinuous refractive index, Inverse Problems, 2017.

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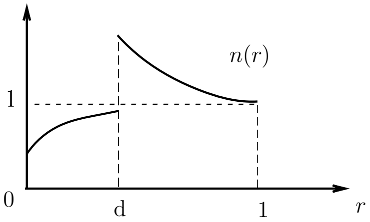

Jump Conditions:

$n(d^+) = a\,n(d^-)$

$n'(d^+) = a^{-1}n'(d^-) + b\,n(d^-)$

$a > 0,\ |a-1| + |b| > 0,$

$d \in (0, 1)$

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Spherically symmetric and discontinuous $n(r)$

Transformed Problem:

▷ Liouville transf.: $z(\xi) := n(r)^{1/4} y_\ell(r),\ \xi(r) := \int_0^r \sqrt{n(\rho)}\, d\rho$

$$\frac{d^2 z(\xi)}{d\xi^2} + \left(k^2 - \frac{\ell(\ell+1)}{\xi^2} - g(\xi)\right) z(\xi) = 0, \quad 0 < \xi \neq \tilde{d}$$

$$g(\xi) = \frac{\ell(\ell+1)}{r^2 n(r)} - \frac{\ell(\ell+1)}{\xi^2} + \frac{n''(r)}{4n(r)^2} - \frac{5n'(r)^2}{16n(r)^3}$$

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where

$$0 < \xi < A := \int_0^1 \sqrt{n(t)}\, dt \ \text{ and } \ \tilde{d} := \int_0^d \sqrt{n(t)}\, dt, \quad \tilde{d} \in (0, A)$$

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▷ z is discontinuous at $\xi = \tilde{d}$, with conditions:

$$z(\tilde{d}^+) = \tilde{a}\, z(\tilde{d}^-), \quad \frac{dz(\tilde{d}^+)}{d\xi} = \tilde{a}^{-1} \frac{dz(\tilde{d}^-)}{d\xi} + \tilde{b}\, n(\tilde{d}^-)$$

where: $|a - 1| + |b| > 0 \ \Rightarrow \ |\tilde{a} - 1| + |\tilde{b}| > 0$

$\tilde{a} = a^{1/4}, \quad \tilde{b} = \frac{1}{4} n(d^-)^{3/4} n(d^+)^{-5/4} b + \frac{1}{4} n'(d^-) n(d^-)^{3/4} n(d^+)^{-9/4} (1 - a^2)$

Spherically symmetric and discontinuous $n(r)$

Asymptotics of the characteristic functions:

▷ for $n(r) \in C^2[0,\infty)$,

$$\text{if } \ell = 0, \quad D_0(k) = \frac{1}{kn(0)^{1/4}} \sin k(1-A) + O\left(\frac{1}{k^2}\right), \; k \to \infty$$

$$\text{if } \ell \geq 1, \quad D_\ell(k) = \frac{1}{kn(0)^{\ell/2+1/4}} \sin k(1-A) + O\left(\frac{\ln k}{k^2}\right), \; k \to \infty$$

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Spherically symmetric and discontinuous $n(r)$

Asymptotics of the characteristic functions:

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Proposition

for $n(r) \in C^2[0,d) \cup C^2(d,1]$:

$$D_0(k) = \frac{1}{kn(0)^{1/4}} \left[\frac{\tilde{a}^2+1}{2\tilde{a}} \sin k(1-A) + \frac{1-\tilde{a}^2}{2\tilde{a}} \sin k\big(1-A+2\tilde{d}\big) \right] + O\Big(\frac{1}{k^2}\Big)$$

$$D_\ell(k) = \frac{1}{kn(0)^{\ell/2+1/4}} \left[\frac{\tilde{a}^2+1}{2\tilde{a}} \sin k(1-A) + (-1)^\ell \frac{1-\tilde{a}^2}{2\tilde{a}} \sin k\big(1-A+2\tilde{d}\big) \right] + O\Big(\frac{\ln k}{k^2}\Big)$$

*proof based on definition of D_ℓ and the Volterra integral equation of $z(\xi)$.

Spherically symmetric and discontinuous $n(r)$

Uniqueness Results:

Theorem (Uniqueness from eigenvalues for $\ell = 0$)

Constants $\tilde{d}, \tilde{a}$ are uniquely determined, if $|\tilde{a} - 1| + |\tilde{b}| > 0$ *and:*
(1). $\tilde{d} \in (0, A)$, *for* $0 < A < 1$
(2). $\tilde{d} \in \left(0, \frac{A-1}{2}\right)$ *or* $\tilde{d} \in \left(\frac{A-1}{2}, A-1\right) \cup (A-1, A)$ *for* $A > 1$

Spherically symmetric and discontinuous $n(r)$

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Theorem (Uniqueness from all eigenvalues for $\ell \geq 0$)

Let $n(r)$ is C^2 or p-w C^2 and satisfies the jump conditions, where $1 - n(r)$ does not change sign. If $n(0)$ is known, then $n(r)$ is uniquely determined by all transmission eigenvalues.

*proofs based on Hadamard's factorization theorem, Müntz theorem and work of Hald[3] for discontinuous Sturm-Liouville problems.

[3] Discontinuous inverse eigenvalue problems, Commun. Pure Appl. Math.,1984

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Inverse Spectral Problem - continuous n

First Results:

Let $A := \int_0^a \sqrt{n(r)}dr$, a the radius, $n(r) > 0, n \in C^1[0,a], n'' \in L^2[0,b]$.

Theorem (McLaughlin and Polyakov (1994))

Assume that for $n_1(r)$ and $n_2(r)$ in the same ball the inf. sequence of the sph. symmetric ITE's are common $\{k_j^2\}_{j=1}^{\infty}$ If also one of the following assumptions holds:

1. $n_1(r) = n_2(r)$ *for* $0 \le b < A$, *with* $0 \le \int_r^b (n_i(r))^{1/2} dr \le (A+b)/2$
2. $n_1(r) = n_2(r)$ *for* $A < b < 3A$ *with* $0 \le \int_r^b (n_i(r))^{1/2} dr \le (3A-b)/2$
3. $3A \le b$

then $n_1(r) = n_2(r)$ *for* $0 \le r < b$.

Inverse Spectral Problem - continuous n

First Results:

Let $A := \int_0^b \sqrt{n(r)}dr$, b the radius, $n(r) > 0, n \in C^1[0,b], n'' \in L^2[0,b]$.

Theorem (McLaughlin and Polyakov (1994))

Assume that for $n_1(r)$ and $n_2(r)$ in the same ball we have an inf. sequence of common ITE's $\{k_j^2\}_{j=1}^{\infty}$ where

1. $\exists$ *an integer* m_0: $|k_j^2| \leq (m+1/2)^2\pi/(A-b)^2$ *for all* $j = 1, ..., m,\ \ m \geq m_0$ *and*
2. *for* $j > m_0$ *all* k_j^2 *are real and* $|k_j^2| \geq (m_0+1/2)^2\pi/(A-b)^2$.

If also one of the following assumptions holds:

1. $n_1(r) = n_2(r)$ *for* $0 \leq b < A$, *with* $0 \leq \int_r^b (n_i(r))^{1/2}dr \leq (A+b)/2$
2. $n_1(r) = n_2(r)$ *for* $A < b < 3A$ *with* $0 \leq \int_r^b (n_i(r))^{1/2}dr \leq (3A-b)/2$
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then $n_1(r) = n_2(r)$ *for* $0 \leq r < b$.

Inverse Spectral Problem - continuous n

Theorem (Aktosun - Gintides - V. Papanicolaou, (2011))

$n(r) > 0, n \in C^1[0, b], n'' \in L^2[0, b]$

(a) If $A < b$, where $A := \int_0^b \sqrt{n(r)}dr$, then the eigenvalues corresponding to spherically symmetric eigenfunctions determine $n(r)$ uniquely.

(b) If $A = b$, then the knowledge of all eigenvalues which are zeros of $\Delta_0(\lambda)$ together with a constant γ determine $n(r)$ uniquely.

Equivalent Eigenvalue Problem for Spherically Symmetric Eigenfunctions

The problem is equivalent to the following eigenvalue problem:

$v'' + \lambda n(r) v = 0, \qquad 0 < r < b$

$$v(0) = 0, \qquad \Delta_0(\lambda) := \frac{\sin(\sqrt{\lambda} b)}{\sqrt{\lambda}} v'(b;\lambda) - \cos(\sqrt{\lambda} b) v(b;\lambda) = 0$$

where $\lambda = k^2$

The zeros λ_n of the entire function $\Delta_0(\lambda)$ are transmission eigenvalues corresponding to spherically symmetric eigenfunctions.

- If $\lambda \in \mathbb{R}$ then $\Delta_0(\lambda) \in \mathbb{R}$.
- The order of $\Delta_0(\lambda)$ is at most $1/2$
- $\Delta_0(0) = 0$

From Hadamard Factorization Theorem

$$\Delta_0(\lambda) = \gamma \lambda^d \prod_{n=1}^{\infty} \left(1 - \frac{\lambda}{\lambda_n}\right), \gamma \in \mathbb{R}, d \geq 1$$

Liouville transform

Auxilliary initial value problem Let $v(r) = v(r;\lambda)$ be the unique solution of the initial-value problem

$$v''(r) + \lambda n(r) v(r) = 0,$$

$$v(0) = 0, \qquad v'(0) = 1.$$

Liouville transformation

$$\zeta := \int_0^r \sqrt{n(\eta)}d\eta, \qquad z(\zeta) = n(r)^{1/4} v(r),$$

transforms the initial-value problem to:

$$z''(\zeta) - p(\zeta)z(\zeta) + \lambda z(\zeta) = 0,$$

$$z(0) = 0, \qquad z'(0) = \frac{1}{n(0)^{1/4}},$$

where $p(\zeta) = \frac{1}{4}\frac{n''(r)}{n(r)^2} - \frac{5}{16}\frac{n'(r)^2}{n(r)^3}$

Asymptotic estimates

There exists a constant $A > 0$ such that

$$\left| v(x;\lambda) - \frac{1}{[n(0)n(x)]^{1/4}\sqrt{\lambda}} \sin\left[\sqrt{\lambda}\int_0^x \sqrt{n(\eta)}d\eta\right]\right|$$

$$\leq \frac{A}{\sqrt{\lambda}} \exp\left[|\Im\{\sqrt{\lambda}\}| \int_0^x \sqrt{n(\eta)}d\eta\right]$$

and

$$\left| v'(x;\lambda) - \left[\frac{n(x)}{n(0)}\right]^{1/4} \cos\left[\sqrt{\lambda}\int_0^x \sqrt{n(\eta)}d\eta\right]\right|$$

$$\leq A \exp\left[|\Im\{\sqrt{\lambda}\}| \int_0^x \sqrt{n(\eta)}d\eta\right]$$

for all $x \in [0, b]$ and all $\lambda \in \mathbb{C}$

Inverse Spectral Problem

Lemma 2.
(a) Assume that $a := \int_0^b \sqrt{n(x)}dx < b$. If $v(x;\lambda)$ satisfies the initial value problem, then

$$v(b;\lambda) = \gamma M(\lambda) \qquad \text{and} \qquad v'(b;\lambda) = \gamma N(\lambda),$$

where $M(\lambda)$ and $N(\lambda)$ are entire functions uniquely determined from $\Delta_0(\lambda)$.
(b) If $a = b$, then $v(b;\lambda) = \frac{\sin\left(b\sqrt{\lambda}\right)}{[n(0)n(b)]^{1/4}\sqrt{\lambda}} + \gamma M(\lambda)$ and $v'(b;\lambda) = \left[\frac{n(b)}{n(0)}\right]^{1/4} \cos\left(b\sqrt{\lambda}\right) + \gamma N(\lambda)$ where $M(\lambda)$, $N(\lambda)$ are as in case (a).

Inverse Spectral Problem

Proof

From the definition of $\Delta_0(\lambda)$ for $\lambda = \pi^2 n^2/b^2$ for $n \in \mathbb{N}$

$$v\left(b; \frac{\pi^2 n^2}{b^2}\right) = (-1)^{n-1}\Delta_0\left(\frac{\pi^2 n^2}{b^2}\right).$$

Similarly, for $\lambda = \pi^2(2n-1)^2/4b^2$, for $n \in \mathbb{N}$, and

$$v'\left(b; \frac{\pi^2(2n-1)^2}{4b^2}\right) = (-1)^{n-1}\frac{\pi(2n-1)}{2b}\Delta_0\left(\frac{\pi^2(2n-1)^2}{4b^2}\right)$$

and application of Lemma 1.

Inverse spectral problem

Lemma 1.

(a) Let $f(\lambda)$ be an entire function such that

$f(\lambda) = \frac{\exp(c|\Im\{\sqrt{\lambda}\}|)}{\sqrt{\lambda}} O(1), \qquad |\lambda| \to \infty$ where $c > 0$

If $f\left(\frac{\pi^2 n^2}{c^2}\right) = 0, \qquad$ for all $n \in \mathbb{N} := \{1, 2, ...\}$ then there is a constant C_1 such that $f(\lambda) = C_1 \frac{\sin\left(c\sqrt{\lambda}\right)}{\sqrt{\lambda}} = C_1 c \prod_{n=1}^{\infty} \left(1 - \frac{c^2 \lambda}{\pi^2 n^2}\right)$

(b) Let $g(\lambda)$ be an entire function such that

$g(\lambda) = \exp(c|\Im\{\sqrt{\lambda}\}|) O(1), \qquad |\lambda| \to \infty$

If $g\left(\frac{\pi^2 (2n-1)^2}{4c^2}\right) = 0, \qquad$ for all $n \in \mathbb{N}$ then there is a constant C_2 such that $g(\lambda) = C_2 \cos\left(c\sqrt{\lambda}\right) = C_2 \prod_{n=1}^{\infty} \left[1 - \frac{4c^2 \lambda}{\pi^2 (2n-1)^2}\right]$