Nonlinear Conservation Laws in Applied Sciences

2017 Summer School in Nonlinear PDE
Lecture 2

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On a fluif-particle interaction model

OUTLINE:

- Modeling
- Energy Estimates
- Notion of Solutions
- Construction of approximate problems
- Main Results: Global-in-time Existence with Large Data
- Challenge: Pass into the limit. Compactness
- Methods: Div-Curl Lemma, Weak continuity of effective viscous pressure, Multipliers technique
- Consequences: Asymptotics, Singular limits

Fluid-Particle Interactions

Fluid-particle interactions arise in many practical applications: biotechnology, medicine, fuel droplets in combustion, sprays etc.
Here, we focus on a particular system derived by formal **asymptotics** from a **mesoscopic description**. This is based on a kinetic equation for the particle distribution of **Fokker-Planck type** coupled to fluid equations.
The coupling between the kinetic and the fluid equations: **friction forces** that the fluid and the particles exert mutually.

The cloud of particles is described by its distribution function $f_\varepsilon(t, x, \xi)$ on phase space, which is the solution to the dimensionless **Vlasov-Fokker-Planck** equation

$$\boxed{\partial_t f_\varepsilon + \frac{1}{\sqrt{\varepsilon}}\Big(\xi \cdot \nabla_x f_\varepsilon - \nabla_x \Phi \cdot \nabla_\xi f_\varepsilon\Big) = \frac{1}{\varepsilon}\,\mathrm{div}_\xi\Big(\big(\xi - \sqrt{\varepsilon} u_\varepsilon\big) f + \nabla_\xi f_\varepsilon\Big).}$$

The friction force is assumed to follow **Stokes law** and thus is proportional to the relative velocity vector, i.e., is proportional to the fluctuations of the microscopic velocity $\xi \in \mathbb{R}^3$ around the fluid velocity field **u**.

The RHS of the moment equation in the Navier-Stokes system takes into account the action of the cloud of particles on the fluid through the **forcing term**

$$\boxed{F_\varepsilon = \int_{\mathbb{R}^3} \left(\frac{\xi}{\sqrt{\varepsilon}} - u_\varepsilon(t,x) \right) f(t,x,\xi)\, d\xi.}$$

The density of the particles $\eta_\varepsilon(t,x)$ is related to the probability distribution function $f_\varepsilon(t,x,\xi)$ through the relation

$$\boxed{\eta_\varepsilon(t,x) = \int_{\mathbb{R}^3} f_\varepsilon(t,x,\xi)\, d\xi.}$$

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0$$

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho(\mathbf{u} \otimes \mathbf{u}) + \nabla_x(p(\varrho) + \eta) - \mu \Delta \mathbf{u} - \lambda \nabla_x \mathrm{div}_x \mathbf{u}$$

$$= -(\eta + \beta \varrho) \nabla \Phi,$$

$$\partial_t \eta + \mathrm{div}(\eta(\mathbf{u} - \nabla \Phi)) - \Delta \eta = 0.$$

$\varrho = \varrho(t, x)$ – total mass density $\qquad t$ – time, $x \in \Omega \subset \mathbb{R}^3$
$u = u(t, x)$ – velocity field
$\eta = \eta(t, x)$ – the density of the particles

$$p(\varrho) = a\varrho^{\gamma} \qquad a > 0, \gamma > 1, \beta \neq 0$$

Φ external potential

$$\mu > 0,\ \lambda + \frac{2}{3}\mu \geq 0 \qquad \text{viscosity parameters}$$

$\beta > 0$ if Ω is unbounded

$\boxed{\text{Boundary Conditions}}$

$$\mathbf{u}|_{\partial\Omega} = \nabla\eta \cdot \nu + \eta\nabla\Phi \cdot \nu = 0 \qquad \text{on} \qquad (0, T) \times \partial\Omega$$

with ν denoting the outer normal vector to the boundary $\partial\Omega$.

$\boxed{\text{Initial Conditions}}$

$(\varrho_0, \mathbf{m}_0, \eta_0)$ such that

$$\begin{cases} \varrho(0,x) = \varrho_0 \in L^\gamma(\Omega) \cap L^1_+(\Omega), \\ (\varrho\mathbf{u})(0,x) = \mathbf{m}_0 \in L^{\frac{6}{5}}(\Omega) \cap L^1(\Omega), \\ \eta(0,x) = \eta_0 \in L^2(\Omega) \cap L^1_+(\Omega). \end{cases}$$

Total Energy

$$E(\eta, \varrho, \mathbf{u})(t) :=$$

$$\int_\Omega [\underbrace{\frac{1}{2}\varrho(t)|\mathbf{u}(t)|^2}_{Kinetic\ \ Energy} + \underbrace{\frac{a}{\gamma-1}\varrho^\gamma(t)}_{Internal\ \ Energy} + \overbrace{(\eta\log\eta)(t)}^{Entropy} + \overbrace{(\beta\varrho+\eta)(t)\Phi}^{Potential\ \ Energy}]dx$$

At the formal level, the total energy can be viewed as a Lyapunov function satisfying the **energy inequality**

$$\frac{dE}{dt} + \int_\Omega \Big[\mu|\nabla\mathbf{u}|^2 + \lambda|\operatorname{div}\mathbf{u}|^2 + |2\nabla\sqrt{\eta} + \sqrt{\eta}\nabla\Phi|^2\Big]dx \le 0.$$

Confinement hypothesis

Given a domain $\Omega \in C^{2,\nu}$, $\nu > 0$, $\Omega \subset \mathbb{R}^3$, and given a bounded from below external potential $\Phi : \Omega \longrightarrow \mathbb{R}_0^+$ satisfying $\inf_{x\in\Omega} \Phi(x) = 0$ we will say that (Ω, Φ) verifies the confinement hypotheses **(HC)** for the two-phase flow system coupled with no-flux boundary conditions whenever:

- If Ω is bounded, Φ is bounded and Lipschitz continuous in $\bar{\Omega}$ and the sub-level sets $[\Phi < k]$ are connected in Ω for any $k > 0$.
- If Ω is unbounded, we assume that $\Phi \in W^{1,\infty}_{loc}(\Omega)$, $\beta > 0$, the sub-level sets $[\Phi < k]$ are connected in Ω for any $k > 0$,

  $$e^{-\Phi/2} \in L^1(\Omega),$$

  and

  $$|\Delta\Phi(x)| \leq c_1|\nabla\Phi(x)| \leq c_2\Phi(x), \ |x| > R > 0$$

Examples

The confinement assumption **(HC)** has physical relevance in our setting as it is verified for several domains Ω with Φ being the gravitational potential. For instance,

1. when $\Omega = \{x \in \mathbb{R}^3 \,|\, (x_1, x_2) \in [a,b]^2,\ x_3 \in [0, H]\}$ and $\Phi(x) = gx_3$, where $\beta = 1 - \frac{\varrho_F}{\varrho_P}$.
2. when $\Omega = \{x \in \mathbb{R}^3 \,|\, (x_1, x_2) \in [a,b]^2,\ x_3 > 0\}$ and $\Phi(x) = gx_3$, where $\beta = 1 - \frac{\varrho_F}{\varrho_P}$ and $\varrho_F < \varrho_P$.
3. when $\Omega = \mathbb{R}^3 \setminus \overline{B(0,R)}$ and $\Phi(x) = g|x|$, where $B(0,R)$ is the ball centered at the origin with radius R and $\beta > 0$.

Here, ϱ_F and ϱ_P are the typical mass density of fluid and particles, respectively. Remark that 1. corresponds to the standard bubbling case in which particles move upwards due to buoyancy.

Derivation of the model

Set up of the model: The particles are viewed as **identical spheres** of radius $R > 0$ with a uniform mass density. Assuming a friction force which is proportional to the velocity difference between the particles and the fluid, on a single particle, the fluid produces a frictional force of

$$F(t, x, \xi) = 6\pi\mu R[\mathbf{u}(t, x) - \xi]$$

where μ is the dynamic viscosity of the fluid. Accordingly, the force exerted by the particles on the fluid is given by the sum

$$6\pi\mu R \int_{\mathbb{R}^3} [\mathbf{u}(t, x) - \xi] f d\xi.$$

The particles move under the influence of Brownian motion, resulting in diffusion in ξ. Here the diffusion coefficient is given by

$$D = \frac{k\theta}{m_P}\frac{6\pi\mu R}{m_P} = \frac{k\theta}{m_P}\frac{9\mu}{2R^2\varrho_P}$$

where

- ϱ_P is the constant mass density of each particle,
- m_P is the total mass of each particle,
- k is the Boltzmann constant, and
- θ is the constant temperature of the system.

Remark.

Here

$$\mathcal{T}_s = \frac{m_P}{6\pi\mu R} = \frac{2R^2\varrho_P}{9\mu}$$

is the natural relaxation time for the kinetic equation usually referred in these applications as the **Stokes settling time**. We consider as usual

$$\mathcal{V}_{th} = \sqrt{\frac{k\theta}{m_P}}$$

as the measure of the *fluctuation of particles velocity*, called their **thermal speed**.

In light of this setup, the equations for the unknowns ϱ, $\mathbf{u}$, and f read

$$\begin{cases} \partial_t \varrho + \mathrm{div}(\varrho \mathbf{u}) = 0 \\ \partial_t(\varrho \mathbf{u}) + \mathrm{div}(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla p(\varrho) = -\beta \varrho \nabla \Phi + \frac{6\pi\mu R}{\varrho_F} \int_{\mathbb{R}^3} (\xi - \mathbf{u}) f \, \mathrm{d}\xi \\ \partial_t f + \xi \cdot \nabla f - \nabla \Phi \cdot \nabla_\xi f = \frac{9\mu}{2R^2 \varrho_P} \mathrm{div}_\xi \left[(\xi - \mathbf{u}) f + \frac{k\theta}{m_P} \nabla_\xi f \right]. \end{cases} \tag{1}$$

Taking unitless parameters and defining

$$\varepsilon = \frac{\mathcal{T}_s}{T}$$

the ratio of the Stokes settling time for microscopic diffusion (denoted $\mathcal{T}_s$) and the characteristic time (denoted T) we obtain:

$$\partial_t \varrho_\varepsilon + \operatorname{div}_x(\varrho_\varepsilon \mathbf{u}_\varepsilon) = 0 \tag{2}$$

$$\partial_t(\varrho_\varepsilon \mathbf{u}_\varepsilon) + \operatorname{div}_x(\varrho_\varepsilon \mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon) + \nabla_x(p(\varrho_\varepsilon)) \tag{3}$$

$$= -\beta \varrho_\varepsilon \nabla_x \Phi + \int_{\mathbb{R}^3} \left(\frac{\xi}{\sqrt{\varepsilon}} - \mathbf{u}_\varepsilon \right) f_\varepsilon \mathrm{d}\xi$$

$$\partial_t f_\varepsilon + \frac{1}{\sqrt{\varepsilon}}(\xi \cdot \nabla_x f_\varepsilon + \nabla_x \Phi \cdot \nabla_\xi f_\varepsilon) = \frac{1}{\varepsilon} \operatorname{div}_\xi[(\xi - \sqrt{\varepsilon}\mathbf{u}_\varepsilon) f_\varepsilon + \nabla_\xi f_\varepsilon]. \tag{4}$$

In the mesoscopic model (2)-(4), ϱ_ε is the fluid density, $\mathbf{u}_\varepsilon$ is the fluid velocity field, and f_ε is the mesoscopic particle density, the three of which are the unknowns. In light of Newton's Third Law, the quantity

$$\int_{\mathbb{R}^3} \left(\frac{\xi}{\sqrt{\varepsilon}} - \mathbf{u}_\varepsilon \right) f_\varepsilon \, \mathrm{d}\xi$$

represents the frictional force exerted by the particles on the fluid. The macroscopic particle density η_ε is related to the mesoscopic particle density f_ε by the relation

$$\eta_\varepsilon(x,t) = \int_{\mathbb{R}^3} f_\varepsilon(x,\xi,t) d\xi. \tag{5}$$

To derive the NSS system from (2)-(4), the asymptotic limit $\varepsilon \to 0$ is taken, representing that the **Stokes settling time** becomes **negligible** compared to the **characteristic time scale** under consideration. Indeed, following the formal analysis in Carrillo and Goudon (2006) and denoting the first and second moments of f_ε as

$$J_\varepsilon(x,t) := \int_{\mathbb{R}^3} \frac{1}{\sqrt{\varepsilon}} \xi f_\varepsilon(x,t,\xi)\,\mathrm{d}\xi$$

and

$$\mathbb{P}_\varepsilon(x,t) := \int_{\mathbb{R}^3} \xi \otimes \xi f_\varepsilon(x,t,\xi)\,\mathrm{d}\xi,$$

respectively, the following equations for the moments are obtained using (4):

$$\partial_t \eta_\varepsilon + \operatorname{div} J_\varepsilon = 0 \tag{6}$$

$$\varepsilon \partial_t J_\varepsilon + \operatorname{div} \mathbb{P}_\varepsilon + \eta_\varepsilon \nabla \Phi = -(J_\varepsilon - \eta_\varepsilon \mathbf{u}_\varepsilon). \tag{7}$$

Taking $\varepsilon \to 0$ and integrating (7) over $\mathbb{R}^3$ yields formally

$$\nabla \eta + \eta \nabla \Phi = -J + \eta \mathbf{u}. \tag{8}$$

Inserting this into (6) and taking the limit yields the Smoluchowski equation.

$$\boxed{\partial_t \eta + \operatorname{div}(\eta(\mathbf{u} - \nabla \Phi)) - \Delta \eta = 0.}$$

Strategy

Variational Formulation

- Derivatives $\sim$ in the sense of distributions
- Equations $\sim$ family of integral identities

Approach

- collect all available **a priori** estimates
- construct a sequence of **approximate problems** whose solutions satisfy these estimate
- show that the sequence of approximate slns **converges** to solution of the **original** problem.

Total Energy

$$E(\eta, \varrho, \mathbf{u})(t) :=$$

$$\int_\Omega [\underbrace{\frac{1}{2}\varrho(t)|\mathbf{u}(t)|^2}_{Kinetic\ \ Energy} + \underbrace{\frac{a}{\gamma - 1}\varrho^\gamma(t)}_{Internal\ \ Energy} + \overbrace{(\eta \log \eta)(t)}^{Entropy} + \overbrace{(\beta \varrho + \eta)(t)\Phi}^{Potential\ \ Energy}] dx$$

At the formal level, the total energy can be viewed as a Lyapunov function satisfying the **energy inequality**

$$\frac{dE}{dt} + \int_\Omega \left[\mu |\nabla \mathbf{u}|^2 + \lambda |\operatorname{div} \mathbf{u}|^2 + |2\nabla\sqrt{\eta} + \sqrt{\eta}\nabla\Phi|^2 \right] dx \leq 0.$$

How do we derive the energy inequality?

We compute the time derivative of each component in the energy.

$$\partial_t \left(\frac{1}{2} \varrho(t) |\mathbf{u}(t)|^2 \right) = ?$$

Multiplying the momentum equation by $\mathbf{u}$:

$$\frac{\partial}{\partial t} \left(\frac{1}{2} \varrho(t) |\mathbf{u}(t)|^2 \right) + \operatorname{div} \left(\frac{1}{2} \varrho(t) |\mathbf{u}(t)|^2 \mathbf{u} \right)$$

$$+ \operatorname{div}(\varrho^\gamma \mathbf{u}) - \varrho^\gamma \operatorname{div} \mathbf{u} + \operatorname{div}(\eta \mathbf{u}) - \eta \operatorname{div} \mathbf{u} =$$

$$\mu \mathbf{u} \Delta \mathbf{u} + \lambda \mathbf{u} \nabla \operatorname{div} \mathbf{u} - (\eta + \beta \varrho) \mathbf{u} \nabla \Phi$$

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$$\frac{\partial}{\partial t} \left(\frac{1}{2} \varrho(t) |\mathbf{u}(t)|^2 \right) + \operatorname{div} \left(\frac{1}{2} \varrho(t) |\mathbf{u}(t)^2 \mathbf{u} \right) + \frac{\mathbf{u}^2}{2} [\partial_t \varrho + \operatorname{div}_x (\varrho \mathbf{u})]$$

$$+ \operatorname{div}(\varrho^\gamma \mathbf{u}) - \varrho^\gamma \operatorname{div} \mathbf{u} + \operatorname{div}(\eta \mathbf{u}) - \eta \operatorname{div} \mathbf{u} =$$

$$\mu \mathbf{u} \Delta \mathbf{u} + \lambda \mathbf{u} \nabla \operatorname{div} \mathbf{u} - (\eta + \beta \varrho) \mathbf{u} \nabla \Phi$$

$$\frac{\partial}{\partial t}\left(\beta(\varrho)\right) = ?$$

Multiplying the continuity equation by $(\beta(\varrho))'$:

$$\frac{\partial}{\partial t}\left(\beta(\varrho)\right) + \operatorname{div}\left(\beta(\varrho)\mathbf{u}\right) - ? = 0$$

$$\operatorname{div}\left(\beta(\varrho)\mathbf{u}\right) = \beta'(\varrho)\nabla\varrho\,\mathbf{u} + \beta(\varrho)\operatorname{div}\mathbf{u}$$

$$\operatorname{div}\left(\beta(\varrho)\mathbf{u}\right) = \beta'(\varrho)\operatorname{div}(\varrho\mathbf{u}) - \beta'(\varrho)\varrho\operatorname{div}\mathbf{u} + \beta(\varrho)\operatorname{div}\mathbf{u}$$

$$\boxed{\frac{\partial}{\partial t}\left(\beta(\varrho)\right) + \operatorname{div}\left(\beta(\varrho)\mathbf{u}\right) + \left[\beta'(\varrho)\varrho - \beta(\varrho)\right]\operatorname{div}\mathbf{u} = 0}$$

$$\frac{\partial}{\partial t}\left(\frac{a\varrho^{\gamma}}{\gamma-1}\right) =?$$

Multiplying the continuity equation by $(\varrho^{\gamma})'$:

$$\boxed{\frac{\partial}{\partial t}(\varrho^{\gamma}) + \operatorname{div}(\varrho^{\gamma}\mathbf{u}) + \varrho^{\gamma}(\gamma-1)\operatorname{div}\mathbf{u} = 0}$$

Indeed:

$$[\beta'(\varrho)\varrho - \beta(\varrho)] = \gamma\varrho^{\gamma-1}\varrho - \varrho^{\gamma} = (\gamma-1)\varrho^{\gamma}$$

Dividing by $(\gamma-1)$:

$$\boxed{\frac{\partial}{\partial t}\left(\frac{\varrho^{\gamma}}{\gamma-1}\right) + \operatorname{div}\left(\frac{\varrho^{\gamma}}{\gamma-1}\mathbf{u}\right) + \varrho^{\gamma}\operatorname{div}\mathbf{u} = 0}$$

$$\partial_t \left(\eta \log \eta\right) = ?$$

Multiplying Smoluchowski equation by $(\eta log \eta)'$:

$$\boxed{\frac{\partial}{\partial t}\left(\eta \log \eta\right) + \operatorname{div}\left(\eta \log \eta \mathbf{u}\right) + \eta \operatorname{div} \mathbf{u} = (log \eta + 1)\Delta\eta}$$

$$\frac{d}{dt}\left(\beta\varrho+\eta\right)(t)\Phi(x) =?$$

$$\boxed{\frac{d}{dt}\left(\beta\varrho+\eta\right)\Phi = \operatorname{div}\left[-(\beta\varrho+\eta)\mathbf{u}\right]\Phi - \operatorname{div}(\eta\nabla\Phi)\Phi - \Delta\eta\Phi.}$$

Adding the above relations, applying integration by parts and using the boundary conditions we obtain:

$$E(\varrho, \mathbf{u}, \eta)(t) + \int_0^t \int_\Omega \left[\mu |\nabla \mathbf{u}|^2 + \lambda |\operatorname{div} \mathbf{u}|^2 + |2\nabla \sqrt{\eta} + \sqrt{\eta} \nabla \Phi|^2 \right] dxds$$

$$= E(\varrho, \mathbf{u}, \eta)(0).$$

with

$$E(t) := \int_\Omega \left[\frac{1}{2} \varrho |\mathbf{u}|^2 + \frac{a}{\gamma - 1} \varrho^\gamma + (\eta \log \eta) + (\beta \varrho + \eta) \Phi \right] dx$$

$$\Downarrow$$

It is reasonable to anticipate that, at least for some sequences $t_n \to \infty$,

$$\eta(t_n) \to \eta_s, \ \ \varrho(t_n) \to \varrho_s, \ \ \varrho\mathbf{u}(t_n) \to 0$$

where η_s, ϱ_s satisfy the stationary problem

$$\nabla(p(\varrho_s) + \eta_s) = -(\eta_s + \beta\varrho_s)\nabla\Phi \ \text{ on } \Omega$$

The energy estimate written in the form

$$E(t) + \int_0^T (\|\nabla\mathbf{u}\|_{L^2}^2 + \|\operatorname{div}\mathbf{u}\|_{L^2}^2)dt + \int_0^T \int_\Omega |2\nabla\sqrt{\eta} + \sqrt{\eta}\nabla\Phi|^2 dt$$

$$\leq E(0)$$

$$|2\nabla\sqrt{\eta_s} + \sqrt{\eta_s}\nabla\Phi|^2 = 0$$

The aim of this paper is to show that, in fact, any weak solution converges to a fixed stationary state as time goes to infinity, more precisely,

$$\varrho(t) \to \varrho_s \quad \text{strongly in} \quad L^\gamma(\Omega),$$
$$\operatorname*{ess\,sup}_{\tau>t} \int_\Omega \varrho(\tau)|\mathbf{u}(\tau)|^2 \to 0,$$
$$\eta(t) \to \eta_s \quad \text{strongly in} \quad L^p(\Omega),$$

as $t \to \infty$ under the **confinement hypothesis** on the domain Ω.

The steady state (ϱ_s, η_s) are determined by

$$\begin{cases} \nabla p(\varrho_s) = -\beta \varrho_s \nabla \Phi, \\ \nabla \eta_s = -\eta_s \nabla \Phi, \end{cases} \qquad \int_\Omega (\varrho_s, \eta_s)\ dx = \int_\Omega (\varrho_0, \eta_0)\ dx,$$

and are given by the formulas

$$\varrho_s = \left(\frac{\gamma - 1}{a\gamma} \left[-\beta \Phi + C_\varrho \right]^+ \right)^{\frac{1}{\gamma - 1}} \qquad \eta_s = C_\eta \exp(-\Phi),$$

where C_η and C_ϱ are uniquely given by the initial masses.

How do we find the appropriate space setting?

Assume smooth solutions. By integrating the continuity equation in space and time:

$$\int_\Omega \varrho dx = \int_\Omega \varrho_0 dx.$$

$$\Downarrow$$

$$\varrho \in L^\infty([0,T]; L^1(\Omega))$$

Bound of Energy implies:

$$\varrho \in L^\infty([0,T]; L^\gamma(\Omega))$$

$$\frac{1}{2}\varrho|\mathbf{u}|^2 \in L^\infty([0,T]; L^1(\Omega)) \Rightarrow \sqrt{\varrho}|\mathbf{u}| \in L^\infty([0,T]; L^2(\Omega))$$

$$\varrho\mathbf{u} \in L^\infty([0,T]; L^{\frac{2\gamma}{\gamma+1}}(\Omega;\mathbb{R}^3))$$

Bound of Energy implies (cont.):

$$\mathbf{u} \in L^2(0, T; W_0^{1,2}(\Omega; \mathbb{R}^3))$$

$$\Downarrow$$

u satisfy the boundary condition in the sense of traces

$$\eta \log \eta \in L^\infty([0, T]; L^1(\Omega))$$

$$\eta \in L^2([0, T]; L^{3/2}(\Omega)) \cap L^1(0, T; W^{1,\frac{3}{2}}(\Omega))$$

The energy inequality yields

$$2\nabla_x \sqrt{\eta} + \sqrt{n}\nabla_x \Phi \in L^2(0,T;L^2(\Omega;\mathbb{R}^3)).$$

Since $\eta \in L^\infty(0,T;L^1(\Omega))$ and $\nabla_x \Phi$ is uniformly bounded

$$\Downarrow$$

$$\nabla_x \sqrt{\eta} \in L^2(0,T;L^2(\Omega;\mathbb{R}^3)),$$

and so

$$\sqrt{\eta} \in L^2(0,T;W^{1,2}(\Omega)) \hookrightarrow L^2(0,T;L^6(\Omega)).$$

Using these estimates, the quantity

$$\nabla_x \eta = 2\sqrt{\eta}\nabla_x\sqrt{\eta},$$

and Hölder's inequality the particle density η satisfies

$$\eta \in L^1(0,T;W^{1,3/2}(\Omega)) \hookrightarrow L^1(0,T;L^3(\Omega)),$$
$$\eta \in L^2(0,T;W^{1,1}(\Omega)) \hookrightarrow L^2(0,T;L^{3/2}(\Omega)).$$

$\boxed{\text{Problem D.}}$

$$
\begin{cases}
\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0 \\
\\
\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho(\mathbf{u} \otimes \mathbf{u}) + \nabla_x(p(\varrho) + \eta) - \mu \Delta \mathbf{u} - \lambda \nabla_x div_x \mathbf{u} \\
\qquad\qquad\qquad\qquad\qquad\qquad = -(\eta + \beta \varrho)\nabla \Phi, \\
\partial_t \eta + \mathrm{div}(\eta(\mathbf{u} - \nabla \Phi)) - \Delta \eta = 0.
\end{cases}
$$

$\boxed{\text{B.C.}}$

$$
\mathbf{u}|_{\partial \Omega} = \nabla \eta \cdot \nu + \eta \nabla \Phi \cdot \nu = 0 \ \text{ on } (0, T) \times \partial \Omega
$$

Strategy

Variational Formulation

- Derivatives $\sim$ in the sense of distributions
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Approach

- collect all available a priori estimates
- construct a sequence of approximate problems whose solutions satisfy these estimate
- show that the sequence of approximate slns converges to solution of the original problem.

Weak solutions

The idea of weak solutions is based on the concept of *generalized derivatives* or distibutions. Classical functions are replaced by *integral averages*

$$f : Q \to R \approx \int_Q f\varphi, \ \varphi \in C_c^\infty(Q).$$

$C_c^\infty(Q)$ denotes the set of infinitely differentiable functions with compact support in Q.
Differential operators D can be conveniently expressed by means of a formal by-parts integration:

$$Df \approx -\int_Q f\, D\varphi, \ \varphi \in C_c^\infty(Q).$$

$$\Downarrow$$

Any (localy) integrable function possesses derivatives of arbitrary order!

Renormalized solutions

Multiplying the continuity equation by $(B'(\varrho))$:

$$\frac{\partial}{\partial t}\left(B(\varrho)\right)+\mathrm{div}\left(B(\varrho)\mathbf{u}\right)+b(\varrho)\,\mathrm{div}_x\,\mathbf{u}=0 \tag{9}$$

where

$$b(z)=B'(z)z-B(z) \tag{10}$$

Definition

We say that ϱ and $\mathbf{u}$ is a renormalized solution of the continuity equation on $(0,T)\times\Omega$ if (9) holds in $\mathcal{D}'((0,T)\times\Omega)$ for any functions

$$B\in C[0,\infty)\cap C^1(0,\infty),\ b\in C[0,\infty)\ \text{bounded on}\ [0,\infty),$$

$$B(0)=b(0)-0$$

satisfying (9)-(10) for all $z>0$.

Free energy solutions

$\{\varrho, \mathbf{u}, \eta\}$ is an admissible **free energy solution** of **Problem D**, supplemented with the initial data $\{\varrho_0, \mathbf{u_0}, \eta_0\}$ provided that

- $\varrho \geq 0$, $\mathbf{u}$ is a **renormalized** solution of the continuity equation,that is,

$$\int_0^T \int_\Omega (\varrho B(\varrho)\partial_t \varphi + \varrho B(\varrho)\mathbf{u} \cdot \nabla_x \varphi - b(\varrho)\,\mathrm{div}\,\mathbf{u}\varphi)\, dxdt$$

$$= - \int_\Omega \varrho_0 B(\varrho_0)\varphi(0, \cdot) dx$$

  holds for any test function $\varphi \in \mathcal{D}([0, T) \times \overline{\Omega})$ and suitable b and B.

- The balance of momentum holds in distributional sense. The velocity field $\mathbf{u}$ belongs to the space $L^2(0, T; W^{1,2}(\Omega; R^3))$, therefore it is legitimate to require $\mathbf{u}$ to satisfy the boundary conditions in the sense of traces.

- $\eta \geq 0$ is a weak solution of the Smoluchowski equation. That is,

$$\int_0^\infty \int_\Omega \eta \partial_t \varphi + \eta \mathbf{u} \cdot \nabla \varphi - \eta \nabla \Phi \cdot \nabla \varphi - \nabla \eta \nabla \varphi dxdt$$

$$= - \int_\Omega \eta_0 \varphi(0, \cdot) dx$$

is satisfied for test functions $\varphi \in \mathcal{D}([0, T) \times \bar{\Omega})$ and any $T > 0$. In particular,

$$\eta \in L^2([0, T]; L^{3/2}(\Omega)) \cap L^1(0, T; W^{1, \frac{3}{2}}(\Omega))$$

- Given the total free-energy of the system by

$$E(\varrho, \mathbf{u}, \eta)(t) := \int_\Omega \left(\frac{1}{2}\varrho|\mathbf{u}|^2 + \frac{a}{\gamma - 1}\varrho^\gamma + \eta \log \eta + (\beta\varrho + \eta)\Phi \right),$$

then $E(\varrho, \mathbf{u}, \eta)(t)$ is finite and bounded by the initial energy of the system

$$E(\varrho, \mathbf{u}, \eta)(t) \leq E(\varrho_0, \mathbf{u}_0, \eta_0) \quad \text{a.e.} \quad t > 0$$

Moreover, the following free energy-dissipation inequality holds

$$\int_0^\infty \int_\Omega \left(\mu|\nabla \mathbf{u}|^2 + \lambda|\operatorname{div} \mathbf{u}|^2 + |2\nabla\sqrt{\eta} + \sqrt{\eta}\nabla\Phi|^2\right)\, dt$$

$$\leq E(\varrho_0, \mathbf{u}_0, \eta_0)$$

Theorem

Let $\Omega \subset \mathbb{R}^3$ bounded domain and (Ω, Φ) satisfy the confinement hypotheses **(HC)**. Then, **Problem D** admits a weak solution $\{\varrho, \mathbf{u}, \eta\}$ on $(0, \infty) \times \Omega$. In addition,

i) the total fluid mass and particle mass given by

$$M_\varrho(t) = \int_\Omega \varrho(t, \cdot)\ dx \qquad \text{and} \qquad M_\eta(t) = \int_\Omega \eta(t, \cdot)\ dx,$$

respectively, are constants of motion.

ii) the density satisfies the higher integrability result

$$\varrho \in L^{\gamma+\Theta}((0, T) \times \Omega), \text{ for any } T > 0,$$

where $\Theta = \min\{\frac{2}{3}\gamma - 1, \frac{1}{4}\}$.

Theorem

Let us assume that (Ω, Φ) satisfy the confinement hypotheses **(HC)**. Then, for any free-energy solution $(\varrho, \mathbf{u}, \eta)$ of the **Problem D**, there exist universal stationary states $\varrho_s(x)$, $\eta_s(x)$, such that

$$\begin{cases} \varrho(t) \to \varrho_s \text{ strongly in } \ L^\gamma(\Omega), \\ \\ \operatorname*{ess\,sup}_{\tau > t} \int_\Omega \varrho(\tau)|\mathbf{u}(\tau)|^2 \, dx \to 0, \\ \\ \eta(t) \to \eta_s \text{ strongly in } \ L^{p_2}(\Omega) \text{ for } p_2 > 1, \\ \end{cases}$$

as $t \to \infty$, where (η_s, ϱ_s) are characterized as the unique free-energy solution of the stationary state problem:

$$\begin{cases} \nabla p(\varrho_s) = -\beta \varrho_s \nabla \Phi, \\ \nabla \eta_s = -\eta_s \nabla \Phi, \end{cases} \qquad \int_\Omega (\varrho_s, \eta_s)\ dx = \int_\Omega (\varrho_0, \eta_0)\ dx,$$

given by the formulas

$$\varrho_s = \left(\frac{\gamma - 1}{a\gamma} [-\beta \Phi + C_\varrho)]^+ \right)^{\frac{1}{\gamma - 1}} \qquad \eta_s = C_\eta \exp(-\Phi),$$

where C_η and C_ϱ are uniquely given by the initial masses.

Remark:

We need to show that the sequences (ϱ_n, η_n) of the time shifts defined as

$$\varrho_n(t,x) := \varrho(t+\tau_n, x), \;\; \tau_n \to \infty,$$
$$\eta_n(t,x) := \eta(t+\tau_n, x), \;\; \tau_n \to \infty,$$

contain subsequences, denoted by the same index w.l.o.g., such that

$$\varrho_n \to \varrho_s \qquad \text{strongly in} \qquad L^1_{loc}((0,1)\times\Omega)$$

and

$$\eta_n \to \eta_s \, \text{strongly in} \;\; L^{p_1}((0,T); L^{p_2})(\Omega) \text{ for some } p_1, p_2 > 1,$$

$\boxed{\textbf{How do we construct suitable approximating problems?}}$

An approximation scheme based on time-descretization

Reference: Carrillo, Karper, Trivisa, Nonlinear Analysis (2011)

Let $\delta > 0$ be fixed. Given a time step $h > 0$, we discretize the time interval $[0, T]$ in terms of the points $t^k = kh$, $k = 0, \ldots, M$, with $Mh = T$. Now, we sequentially determine functions

$$\{\varrho^k_{\delta,h}, \mathbf{u}^k_{\delta,h}, \eta^k_{\delta,h}\} \in \mathcal{W}(\Omega), \quad k = 1, \ldots, M,$$

such that:

- The time discretized continuity equation,

$$d^h_t[\varrho^k_{\delta,h}] + \mathrm{div}(\varrho^k_{\delta,h}\mathbf{u}^k_{\delta,h}) = 0,$$

  holds in the sense of distributions on $\overline{\Omega}$.

- The time discretized momentum equation with artificial pressure,

$$d_t^h[\varrho_{\delta,h}^k \mathbf{u}_{\delta,h}^k] + \operatorname{div}(\varrho_{\delta,h}^k \mathbf{u}_{\delta,h}^k \otimes \mathbf{u}_{\delta,h}^k) - \mu \Delta \mathbf{u}_{\delta,h}^k - \lambda \nabla \operatorname{div} \mathbf{u}_{\delta,h}^k$$

$$+\nabla \left(p_\delta(\varrho_{\delta,h}^k) + \eta_{\delta,h}^k \right) = -(\beta \varrho_{\delta,h}^k + \eta_{\delta,h}^k) \nabla \Phi$$

holds in the sense of distributions on Ω

- The time discretized particle density equation,

$$d_t^h[\eta_{\delta,h}^k] + \operatorname{div} \left(\eta_{\delta,h}^k (\mathbf{u}_{\delta,h}^k - \nabla \Phi) \right) - \Delta \eta_{\delta,h}^k = 0,$$

holds in the sense of distributions on $\overline{\Omega}$.

In the above equations, $d_t^h[\phi^k] = \frac{\phi^k - \phi^{k-1}}{h}$ denotes implicit time stepping.

- There exists an artificial pressure solution ($h \to 0$).
- Vanishing artificial pressure limit ($\delta \to 0$).

An approximating scheme with the aid of Faedo-Galerkin approximations

Here the aproximate solutions are constructed using a three-level approximation scheme. Let X_n for $n \in \mathbb{N}$ denote a family of finite dimensional spaces consisting of smooth vector-valued functions on $\overline{\Omega}$ vanishing of $\partial\Omega$.

Two different types of regularizations are introduced:

1. The ε-regularizations are included to guarantee that certain *a priori* estimates hold true while the energy inequality remains valid at each level of the approximation.
2. The δ-regularization introduces an artificial pressure which is essential in obtaining the convergence result.

Thus, we consider the approximate system:

$$\partial_t \varrho_n + \operatorname{div}(\varrho_n \mathbf{u}_n) = \varepsilon \Delta \varrho_n$$

$$\partial_t \eta_n + \operatorname{div}(\eta_n(\mathbf{u}_n - \nabla \Phi)) = \Delta \eta_n$$

$$\int_\Omega \partial_t(\varrho_n \mathbf{u}_n) \cdot \mathbf{w} \, dx = \int_\Omega \varrho_n \mathbf{u}_n \otimes \mathbf{u}_n : \nabla \mathbf{w} + (a\varrho_n^\gamma + \eta_n + \delta \varrho_n^\alpha) \operatorname{div} \mathbf{w} \, dx$$

$$- \int_\Omega \mathbb{S}(\nabla \mathbf{u}) : \nabla \mathbf{w} + \varepsilon \nabla \varrho_n \cdot \nabla \mathbf{u}_n \cdot \mathbf{w} dx - \int_\Omega (\eta_n \varrho_n + \eta_n) \nabla \Phi \cdot \mathbf{w} \, dx$$

for any $\mathbf{w} \in X_n$, where X_n is a finite dimensional space and α is suitably large exponent.

Boundary conditions

$$\boxed{\nabla_x \varrho \cdot \mathbf{n} = 0, \ \mathbf{u}_n = (\nabla_x \eta_n + \eta_n \nabla_x \Phi) \cdot \mathbf{n} = 0 \text{ on } (0, T) \times \partial \Omega}$$

Here, $\varepsilon, \delta > 0$ are small and α is an appropriate constant. The approximation scheme is also supplemented by the approximate initial data $\{\varrho_{0,\delta}, \mathbf{m}_{0,\delta}, \eta_{0,\delta}\}$. The approximate initial data are modifications of the original initial data in that

- $0 < \delta \leq \varrho_{0,\delta} \leq \delta^{-1/2\alpha}$ for all $x \in \Omega$, $\varrho_{0,\delta} \to \varrho_0$ in $L^\gamma(\Omega)$, and $|\{x \in \Omega | \varrho_{0,\delta}(x) < \varrho_0(x)\}| \to 0$ as $\delta \to 0$.
- $\mathbf{m}_{0,\delta}(x)$ is the same as $\mathbf{m}_0(x)$ unless $\varrho_{0,\delta}(x) < \varrho_0(x)$, in which case $\mathbf{m}_{0,\delta}(x) = 0$.
- $0 < \delta \leq \eta_{0,\delta} \leq \delta^{-1/2\alpha}$ for all $x \in \Omega$, $\eta_{0,\delta} \to \eta_0$ in $L^2(\Omega)$, and $|\{x \in \Omega | \eta_{0,\delta}(x) < \eta_0(x)\}| \to 0$ as $\delta \to 0$.

Motivation

The approximating system is motivated as follows.

- The continuity equation contains the additional Laplacian term $\varepsilon\Delta\varrho$, known as **vanishing viscosity**, in order to increase the regularity of the density ϱ and obtain strong compactness of the density at the first level of the approximation.
- In order to keep the energy estimate satisfied, the $\varepsilon\nabla_x\mathbf{u}\nabla_x\varrho$ term in the modified momentum equation is introduced to balance the vanishing viscosity term.
- Finally, the $\delta\varrho^\alpha$ term in the momentum equation serves to increase the integrability of the pressure during the first two levels of approximation. This is called the **artificial pressure**.

Here, $\varepsilon, \delta > 0$ are small and α is an appropriate constant. The approximation scheme is also supplemented by the approximate initial data $\{\varrho_{0,\delta}, \mathbf{m}_{0,\delta}, \eta_{0,\delta}\}$. The approximate initial data are modifications of the original initial data in that

- $0 < \delta \le \varrho_{0,\delta} \le \delta^{-1/2\alpha}$ for all $x \in \Omega$, $\varrho_{0,\delta} \to \varrho_0$ in $L^\gamma(\Omega)$, and $|\{x \in \Omega | \varrho_{0,\delta}(x) < \varrho_0(x)\}| \to 0$ as $\delta \to 0$.
- $\mathbf{m}_{0,\delta}(x)$ is the same as $\mathbf{m}_0(x)$ unless $\varrho_{0,\delta}(x) < \varrho_0(x)$, in which case $\mathbf{m}_{0,\delta}(x) = 0$.
- $0 < \delta \le \eta_{0,\delta} \le \delta^{-1/2\alpha}$ for all $x \in \Omega$, $\eta_{0,\delta} \to \eta_0$ in $L^2(\Omega)$, and $|\{x \in \Omega | \eta_{0,\delta}(x) < \eta_0(x)\}| \to 0$ as $\delta \to 0$.

In part, these hypotheses ensure that the initial energy

$$E(0) = E_\delta(0) :=$$

$$\int_\Omega \left(\frac{1}{2} \frac{|\mathbf{m}_{\mathbf{0},\delta}|^2}{\varrho_0} + \frac{\delta}{\alpha - 1} \varrho_{0,\delta}^\alpha + \eta_{0,\delta} \log \eta_{0,\delta} + (\beta \varrho_{0,\delta} + \eta_{0,\delta}) \Phi \right) dx,$$

is finite.

Faedo-Galerkin method

The (approximate) initial boundary value problem will be solved via a modified **Faedo-Galerkin method**. We start by introducing a finite-dimensional space

$$X_n = \mathrm{span}\{\pi_j\}_{j=1}^n, \qquad n \in \{1, 2, \dots\}$$

with $\pi_j \in \mathcal{D}(\Omega)^N$ being a set of linearly independent functions which are dense in $C_0^1(\bar{\Omega}, \mathbb{R}^N)$.

The approximate velocities $\mathbf{u}_n \in C([0,T];X_n)$ satisfy a set of integral equations of the form

$$\int_\Omega \varrho \mathbf{u}_n(\tau)\cdot\pi\,dx - \int_\Omega \mathbf{m}_{0,\delta}\cdot\pi =$$

$$\int_0^\tau \int_\Omega (\varrho\mathbf{u}_n \otimes \mathbf{u}_n - \mathbb{S}_n) : \nabla\eta + (p(\varrho) + \eta + \delta\varrho^\beta)\,\mathrm{div}\,\pi\,dxdt$$

$$\int_0^\tau \int_\Omega (\varepsilon\nabla\mathbf{u}_n\nabla\varrho)\cdot\pi\,dxdt,$$

for any test function $\pi \in X_n$, all $\tau \in [0,T]$.

The goal is to seek a **fixed point**

$$\mathbf{u}_n \in C([0,T]; X_n).$$

In order to carry this out, we need information on the mappings assigning each $\mathbf{u}_n$ to unique solutions ϱ, η via the approximate continuity and Smoluchowski equations.

Existence for $\varrho[u_n]$.

Proposition. Let $\Omega \subset \mathbb{R}^N$ be a bounded domain of class $C^{2+\nu}$, $0 < \nu \leq 1$. Suppose that $\varrho_{0,\delta} \in C^{2+\nu}(\bar{\Omega})$ is positive, and satisfies the condition

$$\nabla_x \varrho_{0,\delta} \cdot \mathbf{n} = 0 \text{ on } \partial\Omega.$$

Let $\mathbf{u} \to \varrho[\mathbf{u}]$ assign to any $\mathbf{u} \in C([0,T]; C_0^2(\bar{\Omega};\mathbb{R}^3))$ a unique solution ϱ of the modified fluid density equation. Then this map takes **bounded sets** in the space $C([0,T]; C_0^2(\bar{\Omega};\mathbb{R}^3))$, into **bounded sets** of the space

$$V := \left\{ \begin{array}{l} \partial_t \varrho \in C([0,T]; C^{\nu}(\bar{\Omega}) \\ \varrho \in C([0,T]; C^{2+\nu}(\bar{\Omega}) \end{array} \right.$$

and the map $\mathbf{u} \in C([0,T]; C_0^2(\bar{\Omega};\mathbb{R}^3)) \to \varrho[\mathbf{u}] \in C^1([0,T] \times \bar{\Omega})$ is continuous.

Proposition.

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain of class $C^{2+\nu}, \nu > 0$.

- Then for any $\varrho_{0,\delta}$ and $\mathbf{u} \in C([0,T]; C_0^\nu(\bar{\Omega}))$ there is at most one weak solution $\rho \in L^2(0,T; W^{1,2}(\Omega))$ of the approximate problem.

  If, in addition, $\varrho_{0,\delta}$ belongs to $C^{2+\nu}(\bar{\Omega})$ and satisfies the **(BC)**, then the approximate problem admits a unique classical solution ϱ,

$$\varrho \in C([0,T]; C^{2+\nu}(\bar{\Omega})) \cap C^1([0,T]; C^\nu(\bar{\Omega})).$$

Furthermore,

$$
\begin{aligned}
&(\inf_{\Omega} \varrho_{0,\delta}) \exp\left(- \int_0^\tau \| \operatorname{div} \mathbf{u}_n(t) \|_{L^\infty} \, dt \right) \\
&\leq \varrho(\tau, x) \leq \\
&(\sup_{\Omega} \varrho_{0,\delta}) \exp\left(- \int_0^\tau \| \operatorname{div} \mathbf{u}_n(t) \|_{L^\infty} \, dt \right)
\end{aligned}
\tag{11}
$$

for any $\tau \geq 0$ and any $x \in \Omega$.

- The mapping $\mathbf{u} \to \varrho[\mathbf{u}]$,

$$\varrho : C([0,T]; C_0^2(\bar{\Omega}; \mathbb{R}^N)) \to C([0,T]; C^{2+\nu}(\bar{\Omega}))$$

maps **bounded sets** in $C([0,T]; C_0^2(\bar{\Omega}; \mathbb{R}^N))$ into **bounded sets** in

$$C([0,T]; C^{2+\nu}(\bar{\Omega})) \cap C^1([0,T]; C^{\nu}(\bar{\Omega}))$$

and has the property

$$\|\varrho(\mathbf{u}^1) - \varrho(\mathbf{u}^2)\|_{C([0,T];W^{1,2}(\Omega))} \leq T\, c(r,T) \|\mathbf{u}^1 - \mathbf{u}^2\|_{C([0,T];W_0^{1,2}(\Omega))},$$

for any $\mathbf{u}^1, \mathbf{u}^2$ belonging to the set

$$M_r = \{\mathbf{u} \in C([0,T]; W_0^{1,2}(\Omega)) \mid \|\mathbf{u}(t)\|_{L^\infty(\Omega)} + \|\nabla \mathbf{u}(t)\|_{L^\infty} \leq r, \ \forall t\}.$$

Existence of $\eta[\mathbf{u}_n]$

Proposition. Let $\Omega \subset \mathbb{R}^3$ be a bounded domain of class $C^{2,\nu}, 0 < \nu \leq 1$. Assume that $\eta_{0,\delta} \in C^{0,\nu}(\bar{\Omega})$, and $\mathbf{u} \in C([0,T]; C_0^2(\bar{\Omega}; \mathbb{R}^3))$. Let the compatibility condition

$$(\nabla_x \eta_{0,\delta}(\mathbf{x}) + \eta_{0,\delta}(x)\nabla\Phi(x)) \cdot \mathbf{n}(\mathbf{x}) = 0, \;\; \mathbf{x} \in \partial\Omega$$

be satisfied. Then the Smoluchowski equation has a unique classical solution η such that $\eta \in V$. The solution operator $\mathbf{u} \to \eta[\mathbf{u}]$ assigning to any $\mathbf{u} \in C([0,T]; C_0^2(\bar{\Omega}); \mathbb{R}^3)$ the unique solution of Smoluchowski equation takes **bounded sets** of $C([0,T]; C_0^2(\bar{\Omega}; \mathbb{R}^3))$ into **bounded sets** in V.

Existence of $\mathbf{u}_n$

It is now time to establish the local existence of a solution $\mathbf{u}_n$ on a short interval $[0, T(n)]$ for any fixed $n \in \{1, 2, \dots\}$. Here, we express

$$\mathbf{u}_n(\tau) = \mathcal{M}^{-1}[\varrho(t)] \left(\mathbf{m}^*_{0,\delta} + \int_0^\tau \mathcal{N}[\mathbf{u}_n(t), \varrho(t), \eta(t)]\, dt \right).$$

Now,

$$\mathcal{M}[\varrho] : X_n \to X_n^*, \qquad \mathcal{N}[\varrho] : X_n \to X_n^*$$

are two operators defined by

$$< \mathcal{M}[\varrho]\mathbf{v}, \mathbf{w} >= \int_\Omega \varrho\pi \cdot \mathbf{w}\, dx,$$

$$< \mathcal{N}[\mathbf{u}_n, \varrho, \eta], \pi > = \int_\Omega [\varrho \mathbf{u}_n \otimes \mathbf{u}_n - \mathbb{S}] : \nabla\pi + [p(\varrho) + \eta + \delta\varrho^\alpha] div\pi\, dx$$
$$- \int_\Omega [\varepsilon \nabla \mathbf{u}_n \nabla \varrho + (\beta\varrho + \eta)\nabla_x \Phi] \cdot \pi dx,$$

respectively, with $\varrho = \varrho[\mathbf{u}_n]$, $\eta = \eta[\mathbf{u}_n]$, while X_n^* denotes the dual space of the finite dimensional space X_n and $\mathbf{m}^* \in X_n^*$ is given by

$$< \mathbf{m}_{0,\delta}^*, \pi >= \int_\Omega \mathbf{m}_{0,\delta} \cdot \pi \, dx \qquad \text{for} \qquad \pi \in X_n.$$

Next, observe that the Propositions above now yield that

$$\mathcal{T}[\mathbf{u}] = \mathcal{M}^{-1}[\varrho(t)] \left(\mathbf{m}_{0,\delta}^* + \int_0^\tau \mathcal{N}[\mathbf{u}(t), \varrho(t), \vartheta(t)] \, dt \right)$$

maps compactly the ball

$$B = \{ \mathbf{v} \in C([0,T]; X_n) \, | \, \| \mathbf{v}(t) - \mathbf{u}_{0,\delta,n} \|_{X_n} \leq 1 \} \subset C([0,T]; X_n)$$

into itself at least for a small enough time $T = T(n)$.

By applying the Schauder fixed point theorem and using the estimate

$$\|\mathcal{M}^{-1}[\varrho(\mathbf{u}^1)] - \mathcal{M}^{-1}[\varrho(\mathbf{u}^2)]\|_{\mathcal{L}(X_n^*, X_n)} \leq C(n, n)\|\varrho(\mathbf{u}^1) - \varrho(\mathbf{u}^2)\|_{L^1(\Omega)},$$

we obtain the existence of at least one solution $\mathbf{u}_n$ on the interval $[0, T(n)]$.

Theorem (Schauder fixed point theorem)

Let $\mathcal{B}$ be a closed, convex, bounded subset of a Banach space X, and $\mathcal{T} : \mathcal{B} \to \mathcal{B}$ a compact operator. Then $\mathcal{T}$ has a fixed point.

Indeed, it is easy to check that

$$\sup_{t\in[0,T]} \|\mathcal{T}[\mathbf{u}] - \mathbf{u}_{0,\delta,n}\|_{X_n} \leq c \sup_{t\in[0,T]} (\|\varrho(t) - \varrho_{0,\delta}\|_{L^1(\Omega)} + t).$$

Then, the continuity in time for $\varrho(t)$ implies the right-hand side is made small provided $T = T(n)$ is small. We conclude $\mathcal{T}$ maps $\mathcal{B}$ into itself over a short time interval.

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