Nonlinear Conservation Laws in Applied Sciences

2017 Summer School in Nonlinear PDE
Lecture 3

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Problem D.

$$
\left\{
\begin{aligned}
&\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0 \\
&\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho(\mathbf{u} \otimes \mathbf{u}) + \nabla_x(p(\varrho) + \eta) - \mu \Delta \mathbf{u} - \lambda \nabla_x \mathrm{div}_x \mathbf{u} \\
&\qquad = -(\eta + \beta \varrho) \nabla \Phi, \\
&\partial_t \eta + \mathrm{div}(\eta(\mathbf{u} - \nabla \Phi)) - \Delta \eta = 0.
\end{aligned}
\right.
$$

B.C. $\mathbf{u}|_{\partial\Omega} = \nabla \eta \cdot \nu + \eta \nabla \Phi \cdot \nu = 0 \text{ on } (0, T) \times \partial\Omega$

Strategy

Variational Formulation

- Derivatives $\sim$ in the sense of distributions
- Equations $\sim$ family of integral identities

Approach

- collect all available **a priori** estimates
- construct a sequence of **approximate problems** whose solutions satisfy these estimate
- show that the sequence of approximate solutions **converges** to solution of the original problem.

Weak solutions

The idea of weak solutions is based on the concept of *generalized derivatives* or distibutions. Classical functions are replaced by *integral averages*

$$f : Q \to R \approx \int_Q f \varphi, \ \ \varphi \in C_c^\infty(Q).$$

$C_c^\infty(Q)$ denotes the set of infinitely differentiable functions with compact support in Q.
Differential operators D can be conveniently expressed by means of a formal by-parts integration:

$$Df \approx -\int_Q f\, D\varphi, \ \ \varphi \in C_c^\infty(Q).$$

$$\Downarrow$$

Any (localy) integrable function possesses derivatives of arbitrary order!

Renormalized solutions

Multiplying the continuity equation by $(B'(\varrho))$:

$$\frac{\partial}{\partial t}(B(\varrho)) + \mathrm{div}\,(B(\varrho)\mathbf{u}) + b(\varrho)\,\mathrm{div}_x\,\mathbf{u} = 0 \tag{1}$$

where

$$b(z) = B'(z)z - B(z) \tag{2}$$

Definition

We say that ϱ and $\mathbf{u}$ is a renormalized solution of the continuity equation on $(0,T)\times\Omega$ if (1) holds in $\mathcal{D}'((0,T)\times\Omega)$ for any functions

$$B \in C[0,\infty) \cap C^1(0,\infty),\ b \in C[0,\infty)\ \text{bounded on}\ [0,\infty),$$

$$B(0) = b(0) - 0$$

satisfying (1)-(2) for all $z > 0$.

Free energy solutions

$\{\varrho, \mathbf{u}, \eta\}$ is an admissible **free energy solution** of **Problem D**, supplemented with the initial data $\{\varrho_0, \mathbf{u_0}, \eta_0\}$ provided that

- $\varrho \geq 0$, $\mathbf{u}$ is a **renormalized** solution of the continuity equation,that is,

$$\int_0^T \int_\Omega (\varrho B(\varrho)\partial_t \varphi + \varrho B(\varrho)\mathbf{u} \cdot \nabla_x \varphi - b(\varrho) \operatorname{div} \mathbf{u} \varphi) \, dxdt$$

$$= - \int_\Omega \varrho_0 B(\varrho_0) \varphi(0, \cdot) dx$$

  holds for any test function $\varphi \in \mathcal{D}([0, T) \times \overline{\Omega})$ and suitable b and B.

- The balance of momentum holds in distributional sense. The velocity field $\mathbf{u}$ belongs to the space $L^2(0, T; W^{1,2}(\Omega; R^3))$, therefore it is legitimate to require $\mathbf{u}$ to satisfy the boundary conditions in the sense of traces.

- $\eta \geq 0$ is a weak solution of the Smoluchowski equation. That is,

$$\int_0^\infty \int_\Omega \eta \partial_t \varphi + \eta \mathbf{u} \cdot \nabla \varphi - \eta \nabla \Phi \cdot \nabla \varphi - \nabla \eta \nabla \varphi dxdt$$

$$= - \int_\Omega \eta_0 \varphi(0, \cdot) dx$$

is satisfied for test functions $\varphi \in \mathcal{D}([0, T) \times \bar{\Omega})$ and any $T > 0$. In particular,

$$\eta \in L^2([0, T]; L^{3/2}(\Omega)) \cap L^1(0, T; W^{1, \frac{3}{2}}(\Omega))$$

- Given the total free-energy of the system by

$$E(\varrho, \mathbf{u}, \eta)(t) := \int_\Omega \left(\frac{1}{2}\varrho|\mathbf{u}|^2 + \frac{a}{\gamma - 1}\varrho^\gamma + \eta \log \eta + (\beta\varrho + \eta)\Phi \right),$$

then $E(\varrho, \mathbf{u}, \eta)(t)$ is finite and bounded by the initial energy of the system

$$E(\varrho, \mathbf{u}, \eta)(t) \leq E(\varrho_0, \mathbf{u}_0, \eta_0) \quad \text{a.e.} \quad t > 0$$

Moreover, the following free energy-dissipation inequality holds

$$\int_0^\infty \int_\Omega \left(\mu|\nabla \mathbf{u}|^2 + \lambda|\operatorname{div} \mathbf{u}|^2 + |2\nabla\sqrt{\eta} + \sqrt{\eta}\nabla\Phi|^2 \right) \, dt$$

$$\leq E(\varrho_0, \mathbf{u}_0, \eta_0)$$

Theorem

Let $\Omega \subset \mathbb{R}^3$ bounded domain and (Ω, Φ) satisfy the confinement hypotheses **(HC)**. Then, **Problem D** admits a weak solution $\{\varrho, \mathbf{u}, \eta\}$ on $(0, \infty) \times \Omega$. In addition,

i) the total fluid mass and particle mass given by

$$M_\varrho(t) = \int_\Omega \varrho(t, \cdot)\ dx \qquad \text{and} \qquad M_\eta(t) = \int_\Omega \eta(t, \cdot)\ dx,$$

respectively, are constants of motion.

ii) the density satisfies the higher integrability result

$$\varrho \in L^{\gamma+\Theta}((0, T) \times \Omega), \text{ for any } T > 0,$$

where $\Theta = \min\{\frac{2}{3}\gamma - 1, \frac{1}{4}\}$.

An approximating scheme with the aid of Faedo-Galerkin approximations

Here the aproximate solutions are constructed using a three-level approximation scheme. Let X_n for $n \in \mathbb{N}$ denote a family of finite dimensional spaces consisting of smooth vector-valued functions on $\overline{\Omega}$ vanishing of $\partial\Omega$.

Two different types of regularizations are introduced:

1. The ε-regularizations are included to guarantee that certain *a priori* estimates hold true while the energy inequality remains valid at each level of the approximation.
2. The δ-regularization introduces an artificial pressure which is essential in obtaining the convergence result.

Thus, we consider the approximate system:

$$\partial_t \varrho_n + \operatorname{div}(\varrho_n \mathbf{u}_n) = \varepsilon \Delta \varrho_n$$

$$\partial_t \eta_n + \operatorname{div}(\eta_n(\mathbf{u}_n - \nabla \Phi)) = \Delta \eta_n$$

$$\int_\Omega \partial_t(\varrho_n \mathbf{u}_n) \cdot \mathbf{w} \, dx = \int_\Omega \varrho_n \mathbf{u}_n \otimes \mathbf{u}_n : \nabla \mathbf{w} + (a\varrho_n^\gamma + \eta_n + \delta \varrho_n^\alpha) \operatorname{div} \mathbf{w} \, dx$$

$$- \int_\Omega \mathbb{S}(\nabla \mathbf{u}) : \nabla \mathbf{w} + \varepsilon \nabla \varrho_n \cdot \nabla \mathbf{u}_n \cdot \mathbf{w} dx - \int_\Omega (\eta_n \varrho_n + \eta_n) \nabla \Phi \cdot \mathbf{w} \, dx$$

for any $\mathbf{w} \in X_n$, where X_n is a finite dimensional space and α is suitably large exponent.

Boundary conditions

$$\boxed{\nabla_x \varrho \cdot \mathbf{n} = 0, \ \mathbf{u}_n = (\nabla_x \eta_n + \eta_n \nabla_x \Phi) \cdot \mathbf{n} = 0 \text{ on } (0, T) \times \partial\Omega}$$

Faedo-Galerkin method

The (approximate) initial boundary value problem will be solved via a modified **Faedo-Galerkin method**. We start by introducing a finite-dimensional space

$$X_n = \text{span}\{\pi_j\}_{j=1}^{n}, \qquad n \in \{1, 2, \dots\}$$

with $\pi_j \in \mathcal{D}(\Omega)^N$ being a set of linearly independent functions which are dense in $C_0^1(\bar{\Omega}, \mathbb{R}^N)$.

The approximate velocities $\mathbf{u}_n \in C([0,T]; X_n)$ satisfy a set of integral equations of the form

$$\int_\Omega \varrho \mathbf{u}_n(\tau) \cdot \pi \, dx - \int_\Omega \mathbf{m}_{0,\delta} \cdot \pi =$$

$$\int_0^\tau \int_\Omega (\varrho \mathbf{u}_n \otimes \mathbf{u}_n - \mathbb{S}_n) : \nabla \eta + (p(\varrho) + \eta + \delta \varrho^\beta) \operatorname{div} \pi \, dxdt$$

$$\int_0^\tau \int_\Omega (\varepsilon \nabla \mathbf{u}_n \nabla \varrho) \cdot \pi \, dxdt,$$

for any test function $\pi \in X_n$, all $\tau \in [0,T]$.

The goal is to seek a **fixed point**

$$\mathbf{u}_n \in C([0,T]; X_n).$$

In order to carry this out, we need information on the mappings assigning each $\mathbf{u}_n$ to unique solutions ϱ, η via the approximate continuity and Smoluchowski equations.

Existence for $\varrho[u_n]$.

Proposition. Let $\Omega \subset \mathbb{R}^N$ be a bounded domain of class $C^{2+\nu}$, $0 < \nu \leq 1$. Suppose that $\varrho_{0,\delta} \in C^{2+\nu}(\bar{\Omega})$ is positive, and satisfies the condition

$$\nabla_x \varrho_{0,\delta} \cdot \mathbf{n} = 0 \text{ on } \partial\Omega.$$

Let $\mathbf{u} \to \varrho[\mathbf{u}]$ assign to any $\mathbf{u} \in C([0,T]; C_0^2(\bar{\Omega}; \mathbb{R}^3))$ a unique solution ϱ of the modified fluid density equation. Then this map takes **bounded sets** in the space $C([0,T]; C_0^2(\bar{\Omega}; \mathbb{R}^3))$, into **bounded sets** of the space

$$V := \left\{ \begin{array}{l} \partial_t \varrho \in C([0,T]; C^{\nu}(\bar{\Omega})) \\ \varrho \in C([0,T]; C^{2+\nu}(\bar{\Omega}) \end{array} \right.$$

and the map $\mathbf{u} \in C([0,T]; C_0^2(\bar{\Omega}; \mathbb{R}^3)) \to \varrho[\mathbf{u}] \in C^1([0,T] \times \bar{\Omega})$ is continuous.

Furthermore,

$$
\begin{gathered}
(\inf_{\Omega} \varrho_{0,\delta}) \exp \left(- \int_0^\tau \| \operatorname{div} \mathbf{u}_n(t) \|_{L^\infty} dt \right) \\
\leq \varrho(\tau, x) \leq \\
(\sup_{\Omega} \varrho_{0,\delta}) \exp \left(- \int_0^\tau \| \operatorname{div} \mathbf{u}_n(t) \|_{L^\infty} dt \right)
\end{gathered} \tag{3}
$$

for any $\tau \geq 0$ and any $x \in \Omega$.

- The mapping $\mathbf{u} \to \varrho[\mathbf{u}]$,

$$\varrho : C([0,T]; C_0^2(\bar{\Omega}; \mathbb{R}^N)) \to C([0,T]; C^{2+\nu}(\bar{\Omega}))$$

has the property

$$\|\varrho(\mathbf{u}^1) - \varrho(\mathbf{u}^2)\|_{C([0,T];W^{1,2}(\Omega))} \leq T\, c(r,T) \|\mathbf{u}^1 - \mathbf{u}^2\|_{C([0,T];W_0^{1,2}(\Omega))},$$

for any $\mathbf{u}^1, \mathbf{u}^2$ belonging to the set

$$M_r = \{\mathbf{u} \in C([0,T]; W_0^{1,2}(\Omega)) \mid \|\mathbf{u}(t)\|_{L^\infty(\Omega)} + \|\nabla \mathbf{u}(t)\|_{L^\infty} \leq r, \ \forall t\}.$$

Existence of $\eta[\mathbf{u}_n]$

Proposition. Let $\Omega \subset \mathbb{R}^3$ be a bounded domain of class $C^{2,\nu}, 0 < \nu \leq 1$. Assume that $\eta_{0,\delta} \in C^{0,\nu}(\bar{\Omega})$, and $\mathbf{u} \in C([0,T]; C_0^2(\bar{\Omega}; \mathbb{R}^3))$. Let the compatibility condition

$$(\nabla_x \eta_{0,\delta}(\mathbf{x}) + \eta_{0,\delta}(x)\nabla\Phi(x)) \cdot \mathbf{n}(\mathbf{x}) = 0, \ \mathbf{x} \in \partial\Omega$$

be satisfied. Then the Smoluchowski equation has a unique classical solution η such that $\eta \in V$. The solution operator $\mathbf{u} \to \eta[\mathbf{u}]$ assigning to any $\mathbf{u} \in C([0,T]; C_0^2(\bar{\Omega}); \mathbb{R}^3)$ the unique solution of Smoluchowski equation takes **bounded sets** of $C([0,T]; C_0^2(\bar{\Omega}; \mathbb{R}^3))$ into **bounded sets** in V.

Existence of $\mathbf{u}_n$

It is now time to establish the local existence of a solution $\mathbf{u}_n$ on a short interval $[0, T(n)]$ for any fixed $n \in \{1, 2, \dots\}$. Here, we express

$$\mathbf{u}_n(\tau) = \mathcal{M}^{-1}[\varrho(t)] \left(\mathbf{m}^*_{0,\delta} + \int_0^\tau \mathcal{N}[\mathbf{u}_n(t), \varrho(t), \eta(t)]\, dt \right).$$

Now,

$$\mathcal{M}[\varrho] : X_n \to X_n^*, \qquad \mathcal{N}[\varrho] : X_n \to X_n^*$$

are two operators defined by

$$< \mathcal{M}[\varrho]\mathbf{v}, \mathbf{w} >= \int_\Omega \varrho\pi \cdot \mathbf{w}\, dx,$$

$$< \mathcal{N}[\mathbf{u}_n, \varrho, \eta], \pi > = \int_\Omega [\varrho\mathbf{u}_n \otimes \mathbf{u}_n - \mathbb{S}] : \nabla\pi + [p(\varrho) + \eta + \delta\varrho^\alpha] div \pi\, dx$$
$$- \int_\Omega [\varepsilon\nabla\mathbf{u}_n \nabla\varrho + (\beta\varrho + \eta)\nabla_x\Phi] \cdot \pi dx,$$

respectively, with $\varrho = \varrho[\mathbf{u}_n]$, $\eta = \eta[\mathbf{u}_n]$, while X_n^* denotes the dual space of the finite dimensional space X_n and $\mathbf{m}^* \in X_n^*$ is given by

$$< \mathbf{m}_{0,\delta}^*, \pi >= \int_\Omega \mathbf{m}_{0,\delta} \cdot \pi \, dx \qquad \text{for} \qquad \pi \in X_n.$$

Next, observe that the Propositions above now yield that

$$\mathcal{T}[\mathbf{u}] = \mathcal{M}^{-1}[\varrho(t)] \left(\mathbf{m}_{0,\delta}^* + \int_0^\tau \mathcal{N}[\mathbf{u}(t), \varrho(t), \vartheta(t)] \, dt \right)$$

maps compactly the ball

$$B = \{\mathbf{v} \in C([0,T]; X_n) \, | \, \|\mathbf{v}(t) - \mathbf{u}_{0,\delta,n}\|_{X_n} \leq 1\} \subset C([0,T]; X_n)$$

into itself at least for a small enough time $T = T(n)$.

By applying the Schauder fixed point theorem and using the estimate

$$\|\mathcal{M}^{-1}[\varrho(\mathbf{u}^1)] - \mathcal{M}^{-1}[\varrho(\mathbf{u}^2)]\|_{\mathcal{L}(X_n^*, X_n)} \leq C(n,n)\|\varrho(\mathbf{u}^1) - \varrho(\mathbf{u}^2)\|_{L^1(\Omega)},$$

we obtain the existence of at least one solution $\mathbf{u}_n$ on the interval $[0, T(n)]$.

Theorem (Schauder fixed point theorem)

Let $\mathcal{B}$ be a closed, convex, bounded subset of a Banach space X, and $\mathcal{T} : \mathcal{B} \to \mathcal{B}$ a compact operator. Then $\mathcal{T}$ has a fixed point.

Indeed, it is easy to check that

$$\sup_{t\in[0,T]} \|\mathcal{T}[\mathbf{u}] - \mathbf{u}_{0,\delta,n}\|_{X_n} \leq c \sup_{t\in[0,T]} \left(\|\varrho(t) - \varrho_{0,\delta}\|_{L^1(\Omega)} + t\right).$$

Then, the **continuity in time** for $\varrho(t)$ implies the right-hand side is made small provided $T = T(n)$ is small. We conclude $\mathcal{T}$ maps $\mathcal{B}$ into itself over a short time interval.

Uniform bounds

In order to provide bounds on the various quantities, an **approximate energy balance** is derived by using **u** as a test function in the approximate momentum equation.

$$\Downarrow$$

$$\int_\Omega \left(\frac{1}{2} \varrho_n |\mathbf{u}_n|^2 + \frac{a}{\gamma - 1} \varrho_n^\gamma + \frac{\delta}{\alpha - 1} \varrho_n^\alpha + \eta_n \ln \eta_n + \eta_n \Phi \right) dx$$

$$+ \int_0^T \int_\Omega \left(\mathbb{S}(\nabla \mathbf{u}_n) : \nabla \mathbf{u}_n + |2\nabla \sqrt{\eta_n} + \sqrt{\eta_n} \nabla \Phi|^2 \right) dx\, dt$$

$$+ \varepsilon \int_0^T \int_\Omega |\nabla \varrho_n|^2 (a\gamma \varrho_n^{\gamma-2} + \delta a \varrho_n^{\alpha-2}) dx\, dt$$

$$= \int_\Omega \left(\frac{1}{2} \varrho_{0,\delta} |\mathbf{u}_{0,\delta}|^2 + \frac{a}{\gamma - 1} \varrho_{0,\delta}^\gamma + \frac{\delta}{\alpha - 1} \varrho_{0,\delta}^\alpha + \eta_{0,\delta} \ln \eta_{0,\delta} + \eta_{0,\delta} \Phi \right) dx$$

$$- \beta \int_0^T \int_\Omega \varrho_n \mathbf{u}_n \cdot \nabla \Phi dx\, dt$$

The terms on the left-hand side of the approximate energy balance are all non- negative with the potential exception of $\eta_n \log \eta_n$. We need bounds on the negative contribution $\eta_n \log_- \eta_n$.

Lemma

Suppose Ω is a bounded domain, and $\eta \in L^1_+(\Omega)$. Assume

$$\int_\Omega \eta \log \eta(x) dx \leq C_1 \text{ for some constant } C_1.$$

Then $\eta \log \eta \in L^1(\Omega)$ and

$$\int_\Omega |\eta(x) \log \eta(x)| dx \leq c(C1, |\Omega|).$$

The following bounds are evident from a quick inspection of the approximate energy balance

- $\left\{\sqrt{\varrho}\mathbf{u}\right\}_{n,\varepsilon,\delta} \in_b L^\infty(0,T;L^2(\Omega))$
- $\{\varrho\}_{n,\varepsilon,\delta} \in_b L^\infty(0,T;L^\gamma(\Omega))$
- $\{\eta \ln \eta\}_{n,\varepsilon,\delta} \in_b L^\infty(0,T;L^1(\Omega))$
- $\{\mathbf{u}\}_{n,\varepsilon,\delta} \in_b L^2(0,T;W^{1,2}_0(\Omega))$
- $\left\{\nabla\sqrt{\eta}\right\}_{n,\varepsilon,\delta} \in_b L^2(0,T;L^2(\Omega))$

In addition,

$$\{\eta\}_{n,\varepsilon,\delta} \in_b L^2(0,T;W^{1,\frac{3}{2}}(\Omega)).$$

Convergence of the Approximate Solutions

Now, the goal is to show that the approximate solutions $\{\varrho, \mathbf{u}, \eta\}_{n,\varepsilon,\delta}$ converge to a solution $\{\varrho, \mathbf{u}, \eta\}$ in the sense that we discussed.

The limits are taken as follows.

- Take $n \to \infty$ to obtain $\varrho_n \to \varrho_\varepsilon, \mathbf{u} \to \mathbf{u}_\varepsilon$ and $\eta_n \to \eta_\varepsilon$ in the Faedo-Galerkin approximations.
- Take $\varepsilon \to 0$ to obtain $\varrho_\varepsilon \to \varrho_\delta$, $\mathbf{u}_\varepsilon \to \mathbf{u}_\delta$, and $\eta_\varepsilon \to \eta_\delta$.
- Take $\delta \to 0$ for $\varrho_\delta \to \varrho$, $\mathbf{u}_\delta \to \mathbf{u}$, and $\eta_\delta \to \eta$.

The next step is to show that the above limits can be taken.

Some Remarks on the Existence Theory of Compressible Flow

- Having set the approximation scheme, the existence of solutions are proved locally in time using **a fixed-point argument** and then extended to the full time interval $[0, T]$ using **uniform-in-time estimates**.
- Thereafter, uniform estimates provided by an energy inequality at the Galerkin level allow us to pass to the limit in the first level of the approximation. At each level, **weak lower semicontinuity of the norms** allow us to keep the energy inequality valid.

- In passing to the limit at the second and third levels of the approximation, the linear terms in the NSS system cause no difficulty.
- The nonlinear convective terms

$$(\varrho\mathbf{u}, \eta\mathbf{u}, \varrho\mathbf{u} \otimes \mathbf{u})$$

are handled in a natural way as the NSS system provides estimates on the time derivatives

$$(\partial_t \varrho, \partial_t(\eta\mathbf{u}), \partial_t(\varrho\mathbf{u})).$$

Along with a priori estimates on the velocity

$$\mathbf{u} \in L^2(0, T; H_0^1(\Omega; \mathbb{R}^3)),$$

the passage to the limit in the convective terms follows.

- The difficulty is passing to the limit in the pressure term

$$p(\varrho) = \varrho^{\gamma}.$$

First, the a priori estimates only provide estimates on the pressure in $L^{\infty}(0, T; L^{1}(\Omega))$, which doesnt allow passage to a weak-limit. It is necessary to prove estimates of the form:

$$\int_0^T \int_\Omega \varrho^{\gamma+\omega} dxdt \leq c,$$

where $\omega > 0$ is small. where $\omega > 0$ is small. This type of estimate, first shown by P.L. Lions, allows us to pass to the limit weakly in the pressure to deduce

$$\varrho_{\varepsilon,\delta}^{\gamma} \to \overline{\varrho^{\gamma}},$$

in a suitable Lebesgue space, where the overbar indicates a weak limit. The trick is now to show that in fact $\varrho^{\gamma} = \overline{\varrho^{\gamma}}$ almost everywhere, which requires strong convergence of the fluid density.

Two tools are employed:

- *Renormalization of the continuity equation:* Provided the denisty ϱ is square integrable, we are allowed to conclude that for a suitable function $B(\varrho)$ that

$$\partial_t B(\varrho) + \operatorname{div}_x(B(\varrho)\mathbf{u}) + (B'(\varrho)\varrho - B(\varrho)) \operatorname{div}_x \mathbf{u} = 0$$

  holds in the sense of distributions. This is where Lions requires that $\gamma \geq 9/5$, ensuring square integrability of the density in light of the pressure estimates.
- The weak continuity of the so-called effective viscous pressure, defined as

$$P_{\text{eff}} = \varrho^\gamma - (\lambda + 2\mu) \operatorname{div}_x \mathbf{u}.$$

Choosing the convex function $B(\varrho) = \varrho \log \varrho$ in the renormalized equation for both $\varrho_{\varepsilon,\delta}$ and the limit density ϱ, we can show that

$$0 \leq \int_\Omega \overline{(\varrho \log \varrho)} - \varrho \log \varrho)(t) dx \leq \int_0^t \int_\Omega (\varrho \operatorname{div}_x \mathbf{u} - \overline{\varrho \operatorname{div}_x \mathbf{u}}) dx dt. \tag{4}$$

Remarks:

- The left inequality follows from the convexity of the map $z \to z \log z$. Provided we can show the upper bound of the inequality is non-positive, it follows that

$$\varrho \log \varrho = \overline{\varrho \log \varrho} \text{ a.a.}$$

- To close the argument, the weak continuity of the effective viscous pressure is used to ensure the **non-positivity** of the upper bound in the above relation by transferring information from the monotonicity of the pressure ϱ^γ to the terms $\varrho \operatorname{div}_x \mathbf{u}$.

In particular the result on the effective viscous pressure reads:

$$\lim_{\varepsilon,\delta} \int_0^T \int_\Omega (\varrho_{\varepsilon,\delta}^\gamma - (\lambda + 2\mu)\,\mathrm{div}_x\,\mathbf{u}_{\varepsilon,\delta})\varrho_{\varepsilon,\delta}\,dxdt$$

$$= \int_0^T \int_\Omega (\overline{\varrho^\gamma} - (\lambda + 2\mu)\,\mathrm{div}_x\,\mathbf{u})\varrho\,dxdt.$$

This highly nontrivial equality, along with the monotonicity of the pressure implying

$$\int_0^T \int_\Omega \overline{\varrho^\gamma} \varrho dxdt \leq \liminf_{\varepsilon,\delta} \int_0^T \int_\Omega \varrho_{\varepsilon,\delta}^{\gamma+1} dxdt,$$

allow us to conclude the **nonpositivity of the upper bound** in (4).

Finally, Feireisl (1999) showed that $(\varrho, \mathbf{u})$ is a renormalized solution even if the density is not square integrable, by obtaining estimates on the possible **density oscillations** that can occur in the limit passage. In particular, introducing the oscillations defect measure

$$\boxed{\mathbf{osc}_p[\varrho_\delta \to \varrho](\mathcal{O}) := \sup_{k \geq 1} \left(\limsup_{n \to \infty} \int_{\mathcal{O}} |T_k(\varrho_\delta) - T_k(\varrho)|^p dvdt \right).}$$

Some fundamental techniques

How do we deal with non-linearities?

- **The Div-Curl Lemma:** One of the major discoveries of the compensated compactness theory. [Tartar (1975), Murat]
- **Weak continuity of the effective viscous pressure:** The quantity

$$P_{\text{eff}} = p - (2\mu + \lambda)\,\mathrm{div}\,\mathbf{u}$$

is known as *effective viscous pressure*.

$$\Downarrow$$

STRONG CONVERGENCE OF THE FLUID DENSITY

- **Aubin-Lions lemma**

$$\Downarrow$$

STRONG CONVERGENCE OF THE PARTICLE DENSITY

- **Multipliers Technique:** relies on employing special test functions (solutions of an elliptic problem) in the weak formulation of the momentum equation.

$$\Downarrow$$

INCREASE THE INTEGRABILITY OF THE PRESSURE.

Lemma (**Div-Curl Lemma**)

Let $\Omega \subset \mathbb{R}^N$ be a domain. Let U_n and V_n two sequences of vector functions such that

$$\mathbf{U}_n \to \mathbf{U} \text{ weakly in } L^p(\Omega;\mathbb{R}^N) \tag{5}$$

$$\mathbf{V}_n \to \mathbf{V} \text{ weakly in } L^q(\Omega; R^N). \tag{6}$$

with

$$\frac{1}{p} + \frac{1}{q} \leq 1, \ 1 < p, q < \infty.$$

Furthermore, let

$$\begin{cases} \operatorname{div}_x \mathbf{U}_n = 0 \text{ in } \mathcal{D}'(\Omega) \text{ for } n = 1, 2, \ldots \\ \mathbf{V}_n = \nabla G_n, \text{ with } G_n \text{ bounded in } W^{1,q}(\Omega). \end{cases}$$

Then

$$\mathbf{U}_n \cdot \mathbf{V}_n \to \mathbf{U} \cdot \mathbf{V} \text{ in } \mathcal{D}'(\Omega).$$

Proof of the Simplified Div-Curl Lemma

Up to a subsequence

$$G_n \to G \text{ weakly in } W^{1,q}, \text{ where } \nabla G = \mathbf{V}.$$

Taking arbitrary test function $\varphi \in \mathcal{D}(\Omega)$, we have

$$\int_\Omega \varphi \mathbf{U}_n \cdot \mathbf{V}_n \, dx = \int_\Omega \varphi \mathbf{U}_n \cdot \nabla G_n \, dx = -\int_\Omega G_n \mathbf{U}_n \cdot \nabla \varphi \, dx$$

where the most right integral tends to

$$-\int_\Omega G \mathbf{U} \cdot \nabla \varphi dx = \int_\Omega \varphi \mathbf{U} \cdot \mathbf{V} \, dx.$$

Aubin-Lions lemma

Lemma

Let X_0, X and X_1 be three Banach spaces with $X_0 \subset X \subset X_1$. Suppose that X_0 is compactly embedded in X and that X is continuously embedded in X_1. For $1 \leq p, q \leq +\infty$, let

$$W = \{\mathbf{u} \in L^p([0,T]; X_0),\ \dot{u} \in L^q([0,T]; X_1)\}.$$

(i) If $p < +\infty$, then the embedding of W into $L^p([0,T]; X_1)$ is compact.
(ii) If $p = +\infty$ and $q > 1$, then the embedding of W into $C([0,T]; X)$ is compact.

A consequence of Aubin-Lions lemma

Lemma

Let $X \subset B \subset Y$ be Banach spaces with $X \subset B$ compactly. Then, for $1 \leq p < \infty$, $\{v : v \in L^p(0, T : X), v_t \in L^1(0, T; Y)\}$ is compactly embedded in $L^p(0, T; B)$.

Proof.

For the proof we refer to Simon (1987). □

Thus, applying Aubin-Lions Lemma with $p = 2$, $X = W^{1,\frac{3}{2}}(\Omega)$, $B = L^{3/2}(\Omega)$, and $Y = L^1(\Omega)$ we arrive at

$$\{\eta\}_{n,\varepsilon} \to \eta_\delta \text{ in } L^2(0, T; L^{3/2}(\Omega)). \tag{7}$$

Vanishing viscosity limit

The next step is to let the parameter ε vanish and demonstrate that the solution $(\varrho_\varepsilon, \mathbf{u}_\varepsilon, \eta_\varepsilon)$, constructed in the previous step, converges to $(\varrho_\delta, \mathbf{u}_\delta, \eta_\delta)$.

Challenge: At this stage, we lose control of ϱ_ε in a positive Sobolev space. Demostrating strong compactness for ϱ_ε is the key in order to pass to the limit in the nonlinear terms.

More challenges:

- The energy inequality provides bounds for the pressure $p(\varrho_\varepsilon) + \delta \varrho_\varepsilon$ in $L^1((0,T) \times \Omega)$. This is estimate is not strong enough to prevent concentration phenomena.

The following lemma provides necessary *pressure estimates*.

Lemma

There exists a nonnegative constant c, independent of ε, such that

$$\int_0^T \int_\Omega \varrho_\varepsilon^{\alpha+1} dxdt \le c.$$

Proof.

We postpone the proof for later. □

Strong convergence of the density

There are two key results needed to obtain the strong convergence of the density:

- establishing the weak continuity of the effective viscous pressure, and
- renormalizing the continuity equation both at the level of the approximate solution and the limiting solution. The former is originally due to Lions [54] and asserts that the effective viscous pressure, defined as

$$P_{\text{eff}} = p - (2\mu + \lambda)\,\text{div}_x\,\mathbf{u},$$

satisfies a *weak continuity property* in the sense that its product with another weakly converging sequence converges to the product of the weak limits.

The latter result allows us to deduce that if ϱ satisfies the continuity equation, then so does a suitable nonlinear composition $B(\varrho)$, up to minor modification of the equation.

Riesz operator

To this end, we introduce the *Riesz integral operator,* $\mathcal{R}$:

$$\mathcal{R}_i[v](\mathbf{x}) \equiv (-\Delta)^{-1/2}\partial_{x_i} = c \lim_{\varepsilon \to 0} \int_{\varepsilon \leq |\mathbf{y}| \leq \frac{1}{\varepsilon}} v(\mathbf{x}-\mathbf{y}) \frac{y_i}{|\mathbf{y}^{N+1}|} d\mathbf{y}.$$

or, equivaneIntly, in terms of its Fourier symbol,

$$\mathcal{R}_i(\xi) = \frac{i\xi}{|\xi|}, \ i = 1, \ldots, N.$$

Lemma (Calderon and Zygmund Theorem)

The Riesz operator $\mathcal{R}_i, i = 1, \ldots, N$ defined above is a bounded linear operator on $L^p(\mathbb{R}^N)$ for any $1 < p < \infty$.

Commutators involving the Riesz Operator - Weak convergence

Theorem

Let

$$\mathbf{U}_n \to \mathbf{U} \text{ weakly in } L^p(\Omega; \mathbb{R}^N) \tag{8}$$

$$\mathbf{V}_n \to \mathbf{V} \quad \text{weakly in } L^q(\Omega; R^N). \tag{9}$$

with $\frac{1}{p} + \frac{1}{q} = \frac{1}{s} \leq 1$. *Then,*

$$\mathbf{V}_\varepsilon \cdot \mathcal{R}[\mathbf{U}_\varepsilon] - \mathcal{R}[\mathbf{V}_\varepsilon] \cdot \mathbf{U}_\varepsilon \to \mathbf{V} \cdot \mathcal{R}[\mathbf{U}] - \mathcal{R}[\mathbf{V}] \cdot \mathbf{U}$$

weakly in $L^s(\mathbb{R}^N)$.

Proof of Commutator's Theorem

Writing

$$\mathbf{V}_\varepsilon \cdot \mathcal{R}[\mathbf{U}_\varepsilon] - \mathcal{R}[\mathbf{U}_\varepsilon] \cdot \mathbf{V}_\varepsilon = (\mathbf{V}_\varepsilon - \mathcal{R}[\mathbf{V}_\varepsilon]) \cdot \mathcal{R}[\mathbf{U}_\varepsilon] - (\mathbf{U}_\varepsilon - \mathcal{R}[\mathbf{U}_\varepsilon]) \cdot \mathcal{R}[\mathbf{V}_\varepsilon]$$

we can easily check that

$$\mathrm{div}_x(\mathbf{V}_\varepsilon - \mathcal{R}[\mathbf{V}_\varepsilon]) = \mathrm{div}_x(\mathbf{U}_\varepsilon - \mathcal{R}[\mathbf{U}_\varepsilon]) = 0,$$

while $\mathcal{R}[\mathbf{U}_\varepsilon], \mathcal{R}[\mathbf{V}_\varepsilon]$, are gradient, in particular

$$\mathrm{curl}_x \mathcal{R}[\mathbf{U}_\varepsilon] = \mathrm{curl}_x \mathcal{R}[\mathbf{V}_\varepsilon] = 0.$$

The desired conclusion follows from the Div-Curl Lemma.

Thus, the only terms to consider in the momentum and energy balances are the pressure-related terms. First, using the Bogovskii operator $\mathcal{B}$, analogous to the inverse divergence, the test function $\mathbf{w} := \psi\varphi$ where $\psi \in C_c^\infty(0,T), \varphi := \mathcal{B}[\varrho_\varepsilon - \overline{\varrho}]$ where $\overline{\varrho} := \frac{1}{|\Omega|}\int_\Omega \varrho_\varepsilon \, dx$ in the approximate momentum equation and performing some analysis:

$$\int_0^T \psi \int_\Omega (a\varrho_\varepsilon^\gamma + \eta_\varepsilon + \delta\varrho_\varepsilon^\alpha)\varrho_\varepsilon \, dx \, dt = \int_0^T \psi\overline{\varrho} \int_\Omega a\varrho_\varepsilon^\gamma + \eta_\varepsilon + \delta\varrho_\varepsilon^\alpha \, dx \, dt$$

$$- \int_0^T \psi \int_\Omega \varrho_\varepsilon \mathbf{u}_\varepsilon \cdot \partial_t \varphi dx \, dt - \int_0^T \psi \int_\Omega \varrho_\varepsilon \mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon : \nabla\varphi dx \, dt$$

$$+ \int_0^T \psi \int_\Omega \mathbb{S}(\nabla \mathbf{u}_\varepsilon) : \nabla\varphi dx dt + \int_0^T \psi \int_\Omega (\beta\varrho_\varepsilon + \eta_\varepsilon)\nabla\Phi \cdot \varphi dx \, dt$$

$$- \int_0^T \psi' \int_\Omega \varrho_\varepsilon \mathbf{u}_\varepsilon \cdot \varphi dx \, dt + \varepsilon \int_0^T \psi \int_\Omega \nabla\varrho_\varepsilon \nabla \mathbf{u}_\varepsilon \cdot \varphi dx \, dt.$$

Similarly, the test function

$$\psi\zeta\varphi_2 \text{ with } \varphi_2 := \nabla\Delta^{-1}(\mathbf{1}_\Omega\varrho)$$

is used in the weak limit of the approximate (level-ε) momentum equation, to obtain:

$$\int_0^T \int_\Omega \psi\zeta((\overline{a\varrho^\gamma + \eta + \delta\varrho^\alpha})_\delta \varrho_\delta - \mathbb{S}(\nabla \mathbf{u}_\delta) : \mathcal{RT}(\mathbf{1}_\Omega \varrho_\delta))dxdt$$

$$= \int_0^T \int_\Omega \psi\zeta(\varrho_\delta \mathbf{u} + \delta \cdot \mathcal{RT}(\mathbf{1}_\Omega \varrho_\delta \mathbf{u}_\delta) - (\varrho_\delta \mathbf{u}_\delta \otimes \mathbf{u}_\delta) : \mathcal{RT}(\mathbf{1}_\Omega \varrho_\delta))dxdt$$

$$+ \int_0^T \int_\Omega \psi\zeta(\beta\varrho_\delta + \eta_\delta)\nabla\Phi \cdot \nabla\Delta^{-1}(\mathbf{1}_\Omega \mathbf{u}_\delta)dxdt$$

$$- \int_0^T \int_\Omega \psi(\overline{a\varrho^\gamma + \eta + \delta\varrho^\alpha})_\delta \nabla\zeta \cdot \nabla\Delta^{-1}(\mathbf{1}_\Omega \varrho_\delta)dxdt$$

$$+ \int_0^T \int_\Omega \psi\mathbb{S}(\nabla \mathbf{u}_\delta) : \nabla\zeta \otimes \nabla\Delta^{-1}(\mathbf{1}_\Omega \varrho_\delta)dxdt$$

$$- \int_0^T \int_\Omega \psi(\varrho_\delta \mathbf{u}_\delta \otimes \mathbf{u}_\delta : \nabla\zeta \otimes \nabla\Delta^{-1}(\mathbf{1}_\Omega \varrho_\delta)dxdt$$

$$- \int_0^T \int_\Omega \zeta\varrho_\delta \mathbf{u}_\delta \partial_t \psi \cdot \nabla\Delta^{-1}(\mathbf{1}_\Omega \varrho_\delta)dxdt.$$

From the convergence results stated earlier and the fact that from the theory of elliptic problems the operator $\nabla\Delta^{-1}$ gains a spatial derivative, i.e., due to the embedding $W^{1,\alpha}(\Omega) \hookrightarrow C(\overline{\Omega})$,

$$\nabla\Delta^{-1}(\mathbf{1}_\Omega \varrho_\varepsilon) \to \nabla\Delta^{-1}(\mathbf{1}_\Omega \varrho_\delta)$$

in $C([0,T] \times \overline{\Omega}; \mathbb{R}^3)$. Thus, taking the limit as $\varepsilon \to 0$ in the previous two equations, it follows that

$$\begin{aligned}
&\lim_{\varepsilon\to 0} \int_0^T \int_\Omega \psi\zeta((a\varrho_\varepsilon^\gamma + \eta_\varepsilon + \delta\varrho_\varepsilon^\alpha)\varrho_\varepsilon - \mathbb{S}(\nabla\mathbf{u}_\varepsilon) : \mathcal{R}\mathcal{T}(\mathbf{1}_\Omega\varrho_\varepsilon))dxdt \\
&= \int_0^T \int_\Omega \psi\zeta(\overline{a\varrho^\gamma + \eta + \delta\varrho^\alpha})_\delta \varrho_\delta - \mathbb{S}(\nabla\mathbf{u}_\delta) : \mathcal{R}\mathcal{T}(\mathbf{1}_\Omega\varrho_\delta))dxdt \\
&+ \lim_{\varepsilon\to 0} \int_0^T \int_\Omega \psi\zeta(\varrho_\varepsilon\mathbf{u}_\varepsilon \cdot \mathcal{R}\mathcal{T}(\mathbf{1}_\Omega\varrho_\varepsilon\mathbf{u}_\varepsilon) - (\varrho_\varepsilon\mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon) : \mathcal{R}\mathcal{T}(\mathbf{1}_\Omega\varrho_\varepsilon))dxdt \\
&- \int_0^T \int_\Omega \psi\zeta(\varrho_\delta\mathbf{u}_\delta \cdot \mathcal{R}\mathcal{T}(\mathbf{1}_\Omega\varrho_\delta\mathbf{u}_\delta) - (\varrho_\delta\mathbf{u}_\delta \otimes \mathbf{u}_\delta) : \mathcal{R}\mathcal{T}(\mathbf{1}_\Omega\varrho_\delta))dxdt.
\end{aligned}$$

The goal now is to show that the difference of the last two integrals above vanishes when the limit for ε is taken. This follows from the Commutators's lemma:

Lemma

Let $\mathbf{V}_\varepsilon \to \mathbf{V}$ weakly in $L^p(\mathbb{R}^3;\mathbb{R}^3)$ and $r_\varepsilon \to r$ weakly in $L^q(\mathbb{R}^3)$. Define s such that

$$\frac{1}{p} + \frac{1}{q} = \frac{1}{s} < 1.$$

Then

$$r_\varepsilon \mathcal{R}\mathcal{T}(\mathbf{V}_\varepsilon) - \mathcal{R}\mathcal{T}(r_\varepsilon)\mathbf{V}_\varepsilon \to r\mathcal{R}\mathcal{T}(\mathbf{V}) - \mathcal{R}\mathcal{T}\mathbf{V}$$

weakly in $L^s(\mathbb{R}^3;\mathbb{R}^3)$.

Using the Commutator Lemma and some analysis, the weak compactness identity for the pressure is derived:

$$[\overline{(a\varrho^{\gamma} + \eta + \delta\varrho^{\alpha})\varrho}]_{\delta} - \left(\frac{4}{3}\mu + \lambda\right)(\overline{\varrho \operatorname{div} \mathbf{u}})_{\delta}$$

$$= (\overline{a\varrho^{\gamma} + \eta + \delta\varrho^{\alpha}})_{\delta}\varrho_{\delta} - \left(\frac{4}{3}\mu + \lambda\right)\varrho_{\delta} \operatorname{div} \mathbf{u}_{\delta}.$$

By multiplying the approximate continuity equation by $G'(\varrho_{\varepsilon}) = \varrho_{\varepsilon} \ln \varrho_{\varepsilon}$ noting that G is a smooth convex function, integrating by parts, and taking the weak limit we obtain after some analysis, that

$$(\overline{\varrho \ln \varrho})_{\delta} = \varrho_{\delta} \ln \varrho_{\delta}$$

which since $z \mapsto z \ln z$ is strictly convex, implies that $\varrho_{\varepsilon} \to \varrho_{\delta}$ almost everywhere on $(0, T) \times \Omega$.

Pressure Estimates

Challenge:

The central problem of the mathematical theory of the N-S system is to **control the pressure**. Under the constitutive relations presented here, the pressure p is a priori bounded in $L^1(\Omega)$ uniformly with respect to time.

$$\|p\|_{L^1((0,T;L^1(\Omega))} \leq c_0(E_0, S_0, B_f, T).$$

The non-reflexive Banach space L^1 is not convenient as *bounded sequences are not necessarily weakly pre-compact; more precisely, concentration phenomena may occur to prevent bounded sequences in this space from converging weakly to an integrable function.* Our goal here to find a priori estimates for p in the weakly closed **reflexive space** $L^q((0,T) \times \Omega))$ for a certain $q > 1$.

Idea: Compute p by means of the momentum equation and use the available estimates in order to control the remaining terms.

Local pressure estimates

Idea:

Compute the pressure p in the momentum equation and use the energy estimates already available. Applying the divergence operator to the momentum equation we obtain:

$$\Delta \varrho^{\gamma} = \mathrm{div}_x \, \mathrm{div}_x \, \mathbb{S} - \mathrm{div}_x \, \mathrm{div}_x (\varrho \mathbf{u} \otimes \mathbf{u}) - \Delta \eta - \partial_t \, \mathrm{div}_x (\varrho \mathbf{u})$$

$$+ \mathrm{div}_x (\varrho \beta + \eta) \nabla \Phi \mathbf{v}. \tag{10}$$

Since we already know about the space setting for the quantities $\{\varrho, \varrho \mathbf{u}, \varrho \mathbf{u} \otimes \mathbf{u}, \mathbb{S}\}$, relation (10) can be viewed as an elliptic equation to be resolved with respect to the pressure p to obtain an estimate

$$p \in L^r((0, T) \times \Omega) \text{ for } r > 0.$$

The most problematic term is certainly

$$\boxed{\partial_t \Delta^{-1} \operatorname{div}_x(\varrho \mathbf{u})}$$

for which there are no estimates available.
Here, the idea is to use the fact that ϱ is a **renormalized solution** of the continuity equation. In that case, we can use (10) to obtain

$$\int_0^T \int_\Omega \psi p B(\varrho) \; dxdt$$

$$\approx \text{bounded terms} + \int_0^T \int_\Omega \partial_t \psi (\Delta^{-1} \operatorname{div}_x)[\varrho \mathbf{u}] B(\varrho) \, dxdt$$

$$+ \int_0^T \int_\Omega \partial_t \psi (\Delta^{-1} \operatorname{div}_x)[\varrho \mathbf{u}] \partial_t B(\varrho) dxdt,$$

for any $\psi \in \mathcal{D}(0, T)$, where

$$\partial_t B(\varrho) = -b(\varrho)\,\mathrm{div}_x\,\mathbf{u} - \mathrm{div}_x(B(\varrho)\mathbf{u}).$$

In other words, if we succeed to make this formal procedure rigorous, we get pressure estimates of the form

$$pB(\varrho)\ \text{bounded in}\ L^1_{loc}((0,T)\times\Omega)$$

for a suitable function B.

Riesz operator

To this end, we introduce the *Riesz integral operator*, $\mathcal{R}$:

$$\mathcal{R}_i[v](\mathbf{x}) \equiv (-\Delta)^{-1/2}\partial_{x_i} = c \lim_{\varepsilon\to 0} \int_{\varepsilon\le|\mathbf{y}|\le\frac{1}{\varepsilon}} v(\mathbf{x}-\mathbf{y})\frac{y_i}{|\mathbf{y}^{N+1}}d\mathbf{y}.$$

or, equivaneIntly, in terms of its Fourier symbol,

$$\mathcal{R}_i(\xi) = \frac{i\xi}{|\xi|},\ i = 1,\ldots,N.$$

Lemma (Calderon and Zygmund Theorem)

The Riesz operator $\mathcal{R}_i, i = 1,\ldots,N$ *defined above is a bounded linear operator on* $L^p(\mathbb{R}^N)$ *for any* $1 < p < \infty$.

Lemma

Let $\{\varrho_\delta, \mathbf{u}_\delta, \eta_\delta\}_{\delta>0}$ be a sequence of artificial pressure solutions. Then, there exists a constant $c(T)$, independent of δ, such that

$$\int_0^T \int_\Omega \varrho_\delta^{\gamma+\theta} \, dxdt \leq c(T),$$

where $\Theta = \min\{\frac{2}{3}\gamma - 1, \frac{1}{4}\}$.

If Ω is unbounded, then $\nabla_x \Phi$ is no longer integrable and we cannot simply apply existing results. To prove the bound in this case, let Δ^{-1} be the inverse Laplacian realized using Fourier multipliers.

For each fixed $\delta > 0$, let the test-function $\mathbf{v}_\delta$ be given as

$$\mathbf{v}_\delta = \nabla \Delta^{-1} \varrho_\delta^\theta.$$

Thus,

$$\mathbf{v}_\delta \in_b L^\infty(0,T;W^{1,s}(\Omega)) \cap L^\infty(0,T;L^\infty(\Omega)).$$

Next, since $(\varrho_\delta, \mathbf{u}_\delta)$ is a renormalized solution to the continuity equations, with $B(\varrho_\delta) = \varrho_\delta^\theta$ states

$$\partial_t \varrho_\delta^\theta = -\operatorname{div}(\varrho_\delta^\theta \mathbf{u}) - (\theta - 1)\varrho_\delta^\theta \operatorname{div} \mathbf{u},$$

in the sense of distributions on $(0,T) \times \Omega$. For notational convenience, we observe that

$$\|\partial_t \mathbf{v}_\delta\|_{L^p(0,T;L^q(\Omega))} = \|\nabla \Delta^{-1} \partial_t \varrho_\delta^\theta\|_{L^p(0,T;L^q(\Omega))}$$

$$\leq \|\varrho_\delta^\theta \mathbf{u}_\delta\|_{L^p(0,T;L^q(\Omega))} + \|\varrho^\theta \operatorname{div} \mathbf{u}\|_{L^p(0,T;L^r(\Omega))},$$

for appropriate $1 \leq p, q \leq \infty$ and $r^* = q$.

Next, we apply $\mathbf{v}_\delta$ as test function for the momentum equation to obtain

$$\int_0^T \int_\Omega a\varrho_\delta^{\gamma+\theta}\ dxdt$$

$$= -\int_0^T \int_\Omega (\varrho_\delta \mathbf{u}_\delta)\partial_t \mathbf{v}_\delta + \varrho_\delta \mathbf{u}_\delta \otimes \mathbf{u}_\delta : \nabla \mathbf{v}_\delta\ dxdt$$

$$+ \int_0^T \int_\Omega \mu \nabla \mathbf{u}_\delta \nabla \mathbf{v}_\delta + \lambda \operatorname{div} \mathbf{u}_\delta \operatorname{div} \mathbf{v}_\delta\ dxdt$$

$$- \int_0^T \int_\Omega \eta_\delta \varrho_\delta^\theta - (\varrho_\delta \beta + \eta_\delta)\nabla \Phi_\delta \mathbf{v}_\delta\ dxdt - \int_\Omega \mathbf{m}_0 \mathbf{v}_\delta(0,\cdot)\ dx$$

$$:= I_1 + I_2 + I_3.$$

Artificial pressure limit

At the previous step, the key in renormalizing the continuity equation was using the integrability gain from the artificial pressure to ensure that ϱ_ε was bounded in $L^2(0, T; L^2(\Omega))$. Having lost this integrability through the passage $\delta \to 0$ and since we require that $\gamma > \frac{3}{2}$, we proceed by defining the *oscillation defect measure*. our assumption that the pressure is given by $p(\varrho) = \varrho^\gamma$ for $\gamma > \frac{3}{2}$ The oscillations defect measure is defined as

$$\boxed{\mathbf{osc}_p[\varrho_\delta \to \varrho](\mathcal{O}) := \sup_{k\geq 1} \left(\limsup_{n\to\infty} \int_{\mathcal{O}} |T_k(\varrho_\delta) - T_k(\varrho)|^p dvdt \right).}$$

The oscillation defect measure

$$\boxed{\mathbf{osc}_p[\varrho_\delta \to \varrho](\mathcal{O}) := \sup_{k \geq 1} \left(\limsup_{n \to \infty} \int_{\mathcal{O}} |T_k(\varrho_\delta) - T_k(\varrho)|^p dvdt \right).}$$

The functions T_k are cuttoff functions defined by

$$T_k(z) := kT\left(\frac{z}{k}\right)$$

where T is such that for nonnegative arguments, $T(z) = z$ for $z \in [0, 1]$, $T(z) = 2$ for $z \geq 3$, and a smooth extension is used over the interval $[0, 2]$.

The validity of the weak continuity of the effective viscous pressure implies that the oscillations defect measure is bounded:

$$\mathbf{osc}_{\gamma+1}[\varrho_\delta \to \varrho](\mathcal{O}) \leq c(|\mathcal{O}|).$$

$$\Downarrow$$

$$\varrho_\delta \to \varrho \text{ strongly in } L^1((0,T) \times \Omega).$$

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