Nonlinear Conservation Laws in Applied Sciences

2017 Summer School in Nonlinear PDE
Lecture 4

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$\boxed{\text{Problem D.}}$

$$
\begin{cases}
\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0 \\
\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho(\mathbf{u} \otimes \mathbf{u}) + \nabla_x(p(\varrho) + \eta) - \mu \Delta \mathbf{u} - \lambda \nabla_x \,\mathrm{div}_x\, \mathbf{u} \\
\qquad = -(\eta + \beta \varrho) \nabla \Phi, \\
\partial_t \eta + \mathrm{div}(\eta(\mathbf{u} - \nabla \Phi)) - \Delta \eta = 0.
\end{cases}
$$

$\boxed{\text{B.C.}}$ $$\mathbf{u}|_{\partial\Omega} = \nabla \eta \cdot \nu + \eta \nabla \Phi \cdot \nu = 0 \ \text{ on } (0, T) \times \partial \Omega$$

An approximating scheme with the aid of Faedo-Galerkin approximations

Here the aproximate solutions are constructed using a three-level approximation scheme. Let X_n for $n \in \mathbb{N}$ denote a family of finite dimensional spaces consisting of smooth vector-valued functions on $\overline{\Omega}$ vanishing of $\partial\Omega$.

Two different types of regularizations are introduced:

1. The ε-regularizations are included to guarantee that certain *a priori* estimates hold true while the energy inequality remains valid at each level of the approximation.
2. The δ-regularization introduces an artificial pressure which is essential in obtaining the convergence result.

Thus, we consider the approximate system:

$$\partial_t \varrho_n + \mathrm{div}(\varrho_n \mathbf{u}_n) = \varepsilon \Delta \varrho_n$$

$$\partial_t \eta_n + \mathrm{div}(\eta_n(\mathbf{u}_n - \nabla \Phi)) = \Delta \eta_n$$

$$\int_\Omega \partial_t(\varrho_n \mathbf{u}_n) \cdot \mathbf{w} \, dx = \int_\Omega \varrho_n \mathbf{u}_n \otimes \mathbf{u}_n : \nabla \mathbf{w} + (a\varrho_n^\gamma + \eta_n + \delta \varrho_n^\alpha) \, \mathrm{div}\, \mathbf{w} \, dx$$

$$- \int_\Omega \mathbb{S}(\nabla \mathbf{u}) : \nabla \mathbf{w} + \varepsilon \nabla \varrho_n \cdot \nabla \mathbf{u}_n \cdot \mathbf{w} dx - \int_\Omega (\eta_n \varrho_n + \eta_n) \nabla \Phi \cdot \mathbf{w} \, dx$$

for any $\mathbf{w} \in X_n$, where X_n is a finite dimensional space and α is suitably large exponent.

Boundary conditions

$$\boxed{\nabla_x \varrho \cdot \mathbf{n} = 0, \ \mathbf{u}_n = (\nabla_x \eta_n + \eta_n \nabla_x \Phi) \cdot \mathbf{n} = 0 \text{ on } (0, T) \times \partial \Omega}$$

Faedo-Galerkin method

The (approximate) initial boundary value problem will be solved via a modified **Faedo-Galerkin method**. We start by introducing a finite-dimensional space

$$X_n = \mathrm{span}\{\pi_j\}_{j=1}^{n}, \qquad n \in \{1, 2, \dots\}$$

with $\pi_j \in \mathcal{D}(\Omega)^N$ being a set of linearly independent functions which are dense in $C_0^1(\bar{\Omega}, \mathbb{R}^N)$.

The approximate velocities $\mathbf{u}_n \in C([0,T]; X_n)$ satisfy a set of integral equations of the form

$$\int_\Omega \varrho \mathbf{u}_n(\tau) \cdot \pi \, dx - \int_\Omega \mathbf{m}_{0,\delta} \cdot \pi =$$

$$\int_0^\tau \int_\Omega (\varrho \mathbf{u}_n \otimes \mathbf{u}_n - \mathbb{S}_n) : \nabla \eta + (p(\varrho) + \eta + \delta \varrho^\beta) \operatorname{div} \pi \, dxdt$$

$$\int_0^\tau \int_\Omega (\varepsilon \nabla \mathbf{u}_n \nabla \varrho) \cdot \pi \, dxdt,$$

for any test function $\pi \in X_n$, all $\tau \in [0, T]$.

The limits are taken as follows.

- Take $n \to \infty$ to obtain $\varrho_n \to \varrho_\varepsilon, \mathbf{u} \to \mathbf{u}_\varepsilon$ and $\eta_n \to \eta_\varepsilon$ in the Faedo-Galerkin approximations.
- Take $\varepsilon \to 0$ to obtain $\varrho_\varepsilon \to \varrho_\delta$, $\mathbf{u}_\varepsilon \to \mathbf{u}_\delta$, and $\eta_\varepsilon \to \eta_\delta$.
- Take $\delta \to 0$ for $\varrho_\delta \to \varrho$, $\mathbf{u}_\delta \to \mathbf{u}$, and $\eta_\delta \to \eta$.

The next step is to show that the above limits can be taken.

Some Remarks on the Existence Theory of Compressible Flow

- Having set the approximation scheme, the existence of solutions are proved locally in time using **a fixed-point argument** and then extended to the full time interval $[0, T]$ using **uniform-in-time estimates**.
- Thereafter, uniform estimates provided by an energy inequality at the Galerkin level allow us to pass to the limit in the first level of the approximation. At each level, **weak lower semicontinuity of the norms** allow us to keep the energy inequality valid.

- In passing to the limit at the second and third levels of the approximation, the linear terms in the NSS system cause no difficulty.
- The nonlinear convective terms

$$(\varrho\mathbf{u}, \eta\mathbf{u}, \varrho\mathbf{u} \otimes \mathbf{u})$$

are handled in a natural way as the NSS system provides estimates on the time derivatives

$$(\partial_t \varrho, \partial_t(\eta\mathbf{u}), \partial_t(\varrho\mathbf{u})).$$

Along with a priori estimates on the velocity

$$\mathbf{u} \in L^2(0, T; H_0^1(\Omega; \mathbb{R}^3)),$$

the passage to the limit in the convective terms follows.

- The difficulty is passing to the limit in the pressure term

$$p(\varrho) = \varrho^\gamma.$$

First, the a priori estimates only provide estimates on the pressure in $L^\infty(0, T; L^1(\Omega))$, which doesnt allow passage to a weak-limit. It is necessary to prove estimates of the form:

$$\int_0^T \int_\Omega \varrho^{\gamma+\omega} dxdt \leq c,$$

where $\omega > 0$ is small. where $\omega > 0$ is small. This type of estimate, first shown by P.L. Lions, allows us to pass to the limit weakly in the pressure to deduce

$$\varrho_{\varepsilon,\delta}^\gamma \to \overline{\varrho^\gamma},$$

in a suitable Lebesgue space, where the overbar indicates a weak limit. The trick is now to show that in fact $\varrho^\gamma = \overline{\varrho^\gamma}$ almost everywhere, which requires strong convergence of the fluid density.

Two tools are employed:

- *Renormalization of the continuity equation:* Provided the denisty ϱ is square integrable, we are allowed to conclude that for a suitable function $B(\varrho)$ that

$$\partial_t B(\varrho) + \mathrm{div}_x (B(\varrho)\mathbf{u}) + (B'(\varrho)\varrho - B(\varrho))\, \mathrm{div}_x\, \mathbf{u} = 0$$

  holds in the sense of distributions. This is where Lions requires that $\gamma \geq 9/5$, ensuring square integrability of the density in light of the pressure estimates.
- The weak continuity of the so-called effective viscous pressure, defined as

$$P_{\mathrm{eff}} = \varrho^\gamma - (\lambda + 2\mu)\, \mathrm{div}_x\, \mathbf{u}.$$

Choosing the convex function $B(\varrho) = \varrho \log \varrho$ in the renormalized equation for both $\varrho_{\varepsilon,\delta}$ and the limit density ϱ, we can show that

$$0 \leq \int_\Omega \overline{(\varrho \log \varrho)} - \varrho \log \varrho)(t) dx \leq \int_0^t \int_\Omega (\varrho \operatorname{div}_x \mathbf{u} - \overline{\varrho \operatorname{div}_x \mathbf{u}}) dx dt. \tag{1}$$

Remarks:

- The left inequality follows from the convexity of the map $z \to z \log z$. Provided we can show the upper bound of the inequality is non-positive, it follows that
$$\varrho \log \varrho = \overline{\varrho \log \varrho} \text{ a.a.}$$
- To close the argument, the weak continuity of the effective viscous pressure is used to ensure the **non-positivity** of the upper bound in the above relation by transferring information from the monotonicity of the pressure ϱ^γ to the terms $\varrho \operatorname{div}_x \mathbf{u}$.

In particular the result on the effective viscous pressure reads:

$$\lim_{\varepsilon,\delta} \int_0^T \int_\Omega (\varrho_{\varepsilon,\delta}^\gamma - (\lambda + 2\mu) \operatorname{div}_x \mathbf{u}_{\varepsilon,\delta}) \varrho_{\varepsilon,\delta} dxdt$$

$$= \int_0^T \int_\Omega (\overline{\varrho^\gamma} - (\lambda + 2\mu) \operatorname{div}_x \mathbf{u}) \varrho dxdt.$$

This highly nontrivial equality, along with the monotonicity of the pressure implying

$$\int_0^T \int_\Omega \overline{\varrho^\gamma} \varrho dxdt \leq \liminf_{\varepsilon,\delta} \int_0^T \int_\Omega \varrho_{\varepsilon,\delta}^{\gamma+1} dxdt,$$

allow us to conclude the **nonpositivity of the upper bound** in (1).

Finally, Feireisl (1999) showed that $(\varrho, \mathbf{u})$ is a renormalized solution even if the density is not square integrable, by obtaining estimates on the possible **density oscillations** that can occur in the limit passage. In particular, introducing the oscillations defect measure

$$\boxed{\mathbf{osc}_p[\varrho_\delta \to \varrho](\mathcal{O}) := \sup_{k\geq 1}\left(\limsup_{n\to\infty}\int_{\mathcal{O}}|T_k(\varrho_\delta) - T_k(\varrho)|^p dvdt\right).}$$

Some fundamental techniques

How do we deal with non-linearities?

- **The Div-Curl Lemma:** One of the major discoveries of the compensated compactness theory. [Tartar (1975), Murat]
- **Weak continuity of the effective viscous pressure:** The quantity

$$P_{\text{eff}} = p - (2\mu + \lambda)\,\mathrm{div}\,\mathbf{u}$$

  is known as *effective viscous pressure*.

$$\Downarrow$$

Strong convergence of the fluid density

- **Aubin-Lions lemma**

$$\Downarrow$$

Strong convergence of the particle density

- **Multipliers Technique:** relies on employing special test functions (solutions of an elliptic problem) in the weak formulation of the momentum equation.

$$\Downarrow$$

INCREASE THE INTEGRABILITY OF THE PRESSURE.

Lemma (**Div-Curl Lemma**)

Let $\Omega \subset \mathbb{R}^N$ be a domain. Let U_n and V_n two sequences of vector functions such that

$$\mathbf{U}_n \to \mathbf{U} \text{ weakly in } L^p(\Omega; \mathbb{R}^N) \tag{2}$$

$$\mathbf{V}_n \to \mathbf{V} \text{ weakly in } L^q(\Omega; R^N). \tag{3}$$

with

$$\frac{1}{p} + \frac{1}{q} \leq 1, \; 1 < p, q < \infty.$$

Furthermore, let

$$\begin{cases} \operatorname{div}_x \mathbf{U}_n = 0 \text{ in } \mathcal{D}'(\Omega) \text{ for } n = 1, 2, \dots \\ \mathbf{V}_n = \nabla G_n, \text{ with } G_n \text{ bounded in } W^{1,q}(\Omega). \end{cases}$$

Then

$$\mathbf{U}_n \cdot \mathbf{V}_n \to \mathbf{U} \cdot \mathbf{V} \text{ in} \mathcal{D}'(\Omega).$$

Proof of the Simplified Div-Curl Lemma

Up to a subsequence

$$G_n \to G \text{ weakly in } W^{1,q}, \text{ where } \nabla G = \mathbf{V}.$$

Taking arbitrary test function $\varphi \in \mathcal{D}(\Omega)$, we have

$$\int_\Omega \varphi \mathbf{U}_n \cdot \mathbf{V}_n \, dx = \int_\Omega \varphi \mathbf{U}_n \cdot \nabla G_n \, dx = - \int_\Omega G_n \mathbf{U}_n \cdot \nabla \varphi \, dx$$

where the most right integral tends to

$$- \int_\Omega G \mathbf{U} \cdot \nabla \varphi dx = \int_\Omega \varphi \mathbf{U} \cdot \mathbf{V} \, dx.$$

Aubin-Lions lemma

Lemma

Let X_0, X and X_1 be three Banach spaces with $X_0 \subset X \subset X_1$. Suppose that X_0 is compactly embedded in X and that X is continuously embedded in X_1. For $1 \leq p, q \leq +\infty$, let

$$W = \{\mathbf{u} \in L^p([0,T]; X_0),\ \dot{u} \in L^q([0,T]; X_1)\}.$$

(i) If $p < +\infty$, then the embedding of W into $L^p([0,T]; X_1)$ is compact.
(ii) If $p = +\infty$ and $q > 1$, then the embedding of W into $C([0,T]; X)$ is compact.

A consequence of Aubin-Lions lemma

Lemma

Let $X \subset B \subset Y$ be Banach spaces with $X \subset B$ compactly. Then, for $1 \leq p < \infty$, $\{v : v \in L^p(0, T : X), v_t \in L^1(0, T; Y)\}$ is compactly embedded in $L^p(0, T; B)$.

Proof.

For the proof we refer to Simon (1987). $\square$

Thus, applying Aubin-Lions Lemma with $p = 2$, $X = W^{1,\frac{3}{2}}(\Omega)$, $B = L^{3/2}(\Omega)$, and $Y = L^1(\Omega)$ we arrive at

$$\{\eta\}_{n,\varepsilon} \to \eta_\delta \text{ in } L^2(0, T; L^{3/2}(\Omega)). \tag{4}$$

Vanishing viscosity limit

The next step is to let the parameter ε vanish and demonstrate that the solution $(\varrho_\varepsilon, \mathbf{u}_\varepsilon, \eta_\varepsilon)$, constructed in the previous step, converges to $(\varrho_\delta, \mathbf{u}_\delta, \eta_\delta)$.

Challenge: At this stage, we lose control of ϱ_ε in a positive Sobolev space. Demostrating strong compactness for ϱ_ε is the key in order to pass to the limit in the nonlinear terms.

More challenges:

- The energy inequality provides bounds for the pressure $p(\varrho_\varepsilon) + \delta \varrho_\varepsilon$ in $L^1((0,T) \times \Omega)$. This is estimate is not strong enough to prevent concentration phenomena.

The following lemma provides necessary *pressure estimates*.

Lemma

There exists a nonnegative constant c, *independent of* ε, *such that*

$$\int_0^T \int_\Omega \varrho_\varepsilon^{\alpha+1} dxdt \leq c.$$

Proof.

We postpone the proof for later. $\square$

Strong convergence of the density

There are two key results needed to obtain the strong convergence of the density:

- establishing the weak continuity of the effective viscous pressure, and
- renormalizing the continuity equation both at the level of the approximate solution and the limiting solution. The former is originally due to Lions [54] and asserts that the effective viscous pressure, defined as

$$P_{\text{eff}} = p - (2\mu + \lambda)\, \text{div}_x\, \mathbf{u},$$

  satisfies a *weak continuity property* in the sense that its product with another weakly converging sequence converges to the product of the weak limits.

The latter result allows us to deduce that if ϱ satisfies the continuity equation, then so does a suitable nonlinear composition $B(\varrho)$, up to minor modification of the equation.

Riesz operator

To this end, we introduce the *Riesz integral operator,* $\mathcal{R}$:

$$\mathcal{R}_i[v](\mathbf{x}) \equiv (-\Delta)^{-1/2}\partial_{x_i} = c \lim_{\varepsilon \to 0} \int_{\varepsilon \leq |\mathbf{y}| \leq \frac{1}{\varepsilon}} v(\mathbf{x}-\mathbf{y}) \frac{y_i}{|\mathbf{y}^{N+1}|} d\mathbf{y}.$$

or, equivaneIntly, in terms of its Fourier symbol,

$$\mathcal{R}_i(\xi) = \frac{i\xi}{|\xi|}, \, i = 1, \ldots, N.$$

Lemma (Calderon and Zygmund Theorem)

The Riesz operator $\mathcal{R}_i, i = 1, \ldots, N$ *defined above is a bounded linear operator on* $L^p(\mathbb{R}^N)$ *for any* $1 < p < \infty$.

Commutators involving the Riesz Operator - Weak convergence

Theorem

Let

$$\mathbf{U}_n \to \mathbf{U} \ \textit{weakly in}\ L^p(\Omega;\mathbb{R}^N) \tag{5}$$
$$\mathbf{V}_n \to \mathbf{V} \ \ \textit{weakly in}\ L^q(\Omega;R^N). \tag{6}$$

with $\frac{1}{p}+\frac{1}{q}=\frac{1}{s}\leq 1$. *Then,*

$$\mathbf{V}_\varepsilon \cdot \mathcal{R}[\mathbf{U}_\varepsilon] - \mathcal{R}[\mathbf{V}_\varepsilon]\cdot \mathbf{U}_\varepsilon \to \mathbf{V}\cdot\mathcal{R}[\mathbf{U}] - \mathcal{R}[\mathbf{V}]\cdot\mathbf{U}$$

weakly in $L^s(\mathbb{R}^N)$.

Proof of Commutator's Theorem

Writing

$$\mathbf{V}_\varepsilon \cdot \mathcal{R}[\mathbf{U}_\varepsilon] - \mathcal{R}[\mathbf{U}_\varepsilon] \cdot \mathbf{V}_\varepsilon = (\mathbf{V}_\varepsilon - \mathcal{R}[\mathbf{V}_\varepsilon]) \cdot \mathcal{R}[\mathbf{U}_\varepsilon] - (\mathbf{U}_\varepsilon - \mathcal{R}[\mathbf{U}_\varepsilon]) \cdot \mathcal{R}[\mathbf{V}_\varepsilon]$$

we can easily check that

$$\mathrm{div}_x(\mathbf{V}_\varepsilon - \mathcal{R}[\mathbf{V}_\varepsilon]) = \mathrm{div}_x(\mathbf{U}_\varepsilon - \mathcal{R}[\mathbf{U}_\varepsilon]) = 0,$$

while $\mathcal{R}[\mathbf{U}_\varepsilon], \mathcal{R}[\mathbf{V}_\varepsilon]$, are gradient, in particular

$$\mathrm{curl}_x \mathcal{R}[\mathbf{U}_\varepsilon] = \mathrm{curl}_x \mathcal{R}[\mathbf{V}_\varepsilon] = 0.$$

The desired conclusion follows from the Div-Curl Lemma.

Thus, the only terms to consider in the momentum and energy balances are the pressure-related terms. First, using the Bogovskii operator $\mathcal{B}$, analogous to the inverse divergence, the test function $\mathbf{w} := \psi\varphi$ where $\psi \in C_c^\infty(0, T), \varphi := \mathcal{B}[\varrho_\varepsilon - \overline{\varrho}]$ where $\overline{\varrho} := \frac{1}{|\Omega|}\int_\Omega \varrho_\varepsilon \, dx$ in the approximate momentum equation and performing some analysis:

$$\int_0^T \psi \int_\Omega (a\varrho_\varepsilon^\gamma + \eta_\varepsilon + \delta\varrho_\varepsilon^\alpha)\varrho_\varepsilon \, dx \, dt = \int_0^T \psi\overline{\varrho} \int_\Omega a\varrho_\varepsilon^\gamma + \eta_\varepsilon + \delta\varrho_\varepsilon^\alpha \, dx \, dt$$

$$- \int_0^T \psi \int_\Omega \varrho_\varepsilon \mathbf{u}_\varepsilon \cdot \partial_t \varphi dx \, dt - \int_0^T \psi \int_\Omega \varrho_\varepsilon \mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon : \nabla\varphi dx \, dt$$

$$+ \int_0^T \psi \int_\Omega \mathbb{S}(\nabla \mathbf{u}_\varepsilon) : \nabla\varphi dxdt + \int_0^T \psi \int_\Omega (\beta\varrho_\varepsilon + \eta_\varepsilon)\nabla\Phi \cdot \varphi dx \, dt$$

$$- \int_0^T \psi' \int_\Omega \varrho_\varepsilon \mathbf{u}_\varepsilon \cdot \varphi dx \, dt + \varepsilon \int_0^T \psi \int_\Omega \nabla\varrho_\varepsilon \nabla \mathbf{u}_\varepsilon \cdot \varphi dx \, dt.$$

Similarly, the test function

$$\psi\zeta\varphi_2 \text{ with } \varphi_2 := \nabla\Delta^{-1}(\mathbf{1}_{\Omega}\varrho)$$

is used in the weak limit of the approximate (level-ε) momentum equation, to obtain:

$$\int_0^T \int_\Omega \psi\zeta((\overline{a\varrho^\gamma + \eta + \delta\varrho^\alpha})_\delta \varrho_\delta - \mathbb{S}(\nabla \mathbf{u}_\delta) : \mathcal{RT}(\mathbf{1}_\Omega \varrho_\delta)) dxdt$$

$$= \int_0^T \int_\Omega \psi\zeta(\varrho_\delta \mathbf{u} + \delta \cdot \mathcal{RT}(\mathbf{1}_\Omega \varrho_\delta \mathbf{u}_\delta) - (\varrho_\delta \mathbf{u}_\delta \otimes \mathbf{u}_\delta) : \mathcal{RT}(\mathbf{1}_\Omega \varrho_\delta)) dxdt$$

$$+ \int_0^T \int_\Omega \psi\zeta(\beta\varrho_\delta + \eta_\delta) \nabla\Phi \cdot \nabla\Delta^{-1}(\mathbf{1}_\Omega \mathbf{u}_\delta) dxdt$$

$$- \int_0^T \int_\Omega \psi(\overline{a\varrho^\gamma + \eta + \delta\varrho^\alpha})_\delta \nabla\zeta \cdot \nabla\Delta^{-1}(\mathbf{1}_\Omega \varrho_\delta) dxdt$$

$$+ \int_0^T \int_\Omega \psi\mathbb{S}(\nabla \mathbf{u}_\delta) : \nabla\zeta \otimes \nabla\Delta^{-1}(\mathbf{1}_\Omega \varrho_\delta) dxdt$$

$$- \int_0^T \int_\Omega \psi(\varrho_\delta \mathbf{u}_\delta \otimes \mathbf{u}_\delta : \nabla\zeta \otimes \nabla\Delta^{-1}(\mathbf{1}_\Omega \varrho_\delta) dxdt$$

$$- \int_0^T \int_\Omega \zeta\varrho_\delta \mathbf{u}_\delta \partial_t \psi \cdot \nabla\Delta^{-1}(\mathbf{1}_\Omega \varrho_\delta) dxdt.$$

From the convergence results stated earlier and the fact that from the theory of elliptic problems the operator $\nabla\Delta^{-1}$ gains a spatial derivative, i.e., due to the embedding $W^{1,\alpha}(\Omega) \hookrightarrow C(\overline{\Omega})$,

$$\nabla\Delta^{-1}(\mathbf{1}_\Omega \varrho_\varepsilon) \to \nabla\Delta^{-1}(\mathbf{1}_\Omega \varrho_\delta)$$

in $C([0,T]\times\overline{\Omega};\mathbb{R}^3)$. Thus, taking the limit as $\varepsilon \to 0$ in the previous two equations, it follows that

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\int_0^T\int_\Omega \psi\zeta((a\varrho_\varepsilon^\gamma + \eta_\varepsilon + \delta\varrho_\varepsilon^\alpha)\varrho_\varepsilon - \mathbb{S}(\nabla\mathbf{u}_\varepsilon) : \mathcal{R}\mathcal{T}(\mathbf{1}_\Omega\varrho_\varepsilon))dxdt \\
&= \int_0^T\int_\Omega \psi\zeta(\overline{a\varrho^\gamma + \eta + \delta\varrho^\alpha})_\delta\varrho_\delta - \mathbb{S}(\nabla\mathbf{u}_\delta) : \mathcal{R}\mathcal{T}(\mathbf{1}_\Omega\varrho_\delta))dxdt \\
&+ \lim_{\varepsilon\to 0}\int_0^T\int_\Omega \psi\zeta(\varrho_\varepsilon\mathbf{u}_\varepsilon\cdot\mathcal{R}\mathcal{T}(\mathbf{1}_\Omega\varrho_\varepsilon\mathbf{u}_\varepsilon) - (\varrho_\varepsilon\mathbf{u}_\varepsilon\otimes\mathbf{u}_\varepsilon) : \mathcal{R}\mathcal{T}(\mathbf{1}_\Omega\varrho_\varepsilon))dxdt \\
&- \int_0^T\int_\Omega \psi\zeta(\varrho_\delta\mathbf{u}_\delta\cdot\mathcal{R}\mathcal{T}(\mathbf{1}_\Omega\varrho_\delta\mathbf{u}_\delta) - (\varrho_\delta\mathbf{u}_\delta\otimes\mathbf{u}_\delta) : \mathcal{R}\mathcal{T}(\mathbf{1}_\Omega\varrho_\delta))dxdt.
\end{aligned}$$

The goal now is to show that the difference of the last two integrals above vanishes when the limit for ε is taken. This follows from the Commutators's lemma:

Lemma

Let $\mathbf{V}_\varepsilon \to \mathbf{V}$ *weakly in* $L^p(\mathbb{R}^3;\mathbb{R}^3)$ *and* $r_\varepsilon \to r$ *weakly in* $L^q(\mathbb{R}^3)$. *Define* s *such that*

$$\frac{1}{p} + \frac{1}{q} = \frac{1}{s} < 1.$$

Then

$$r_\varepsilon \mathcal{RT}(\mathbf{V}_\varepsilon) - \mathcal{RT}(r_\varepsilon)\mathbf{V}_\varepsilon \to r\mathcal{RT}(\mathbf{V}) - \mathcal{RT}\mathbf{V}$$

weakly in $L^s(\mathbb{R}^3;\mathbb{R}^3)$.

Using the Commutator Lemma and some analysis, the weak compactness identity for the pressure is derived:

$$[\overline{(a\varrho^\gamma + \eta + \delta\varrho^\alpha)\varrho}]_\delta - \left(\frac{4}{3}\mu + \lambda\right)(\overline{\varrho \operatorname{div} \mathbf{u}})_\delta$$

$$= (\overline{a\varrho^\gamma + \eta + \delta\varrho^\alpha})_\delta \varrho_\delta - \left(\frac{4}{3}\mu + \lambda\right)\varrho_\delta \operatorname{div} \mathbf{u}_\delta .$$

By multiplying the approximate continuity equation by $G'(\varrho_\varepsilon) = \varrho_\varepsilon \ln \varrho_\varepsilon$ noting that G is a smooth convex function, integrating by parts, and taking the weak limit we obtain after some analysis, that

$$(\overline{\varrho \ln \varrho})_\delta = \varrho_\delta \ln \varrho_\delta$$

which since $z \mapsto z \ln z$ is strictly convex, implies that $\varrho_\varepsilon \to \varrho_\delta$ almost everywhere on $(0, T) \times \Omega$.

Pressure Estimates

Challenge:

The central problem of the mathematical theory of the N-S system is to **control the pressure**. Under the constitutive relations presented here, the pressure p is a priori bounded in $L^1(\Omega)$ uniformly with respect to time.

$$\|p\|_{L^1((0,T;L^1(\Omega))} \leq c_0(E_0, S_0, B_f, T).$$

The non-reflexive Banach space L^1 is not convenient as *bounded sequences are not necessarily weakly pre-compact; more precisely, concentration phenomena may occur to prevent bounded sequences in this space from converging weakly to an integrable function.* Our goal here to find a priori estimates for p in the weakly closed **reflexive space** $L^q((0,T)\times\Omega))$ for a certain $q > 1$.

Idea: Compute p by means of the momentum equation and use the available estimates in order to control the remaining terms.

Local pressure estimates

Idea:

Compute the pressure p in the momentum equation and use the energy estimates already available. Applying the divergence operator to the momentum equation we obtain:

$$\Delta \varrho^{\gamma} = \mathrm{div}_x \, \mathrm{div}_x \, \mathbb{S} - \mathrm{div}_x \, \mathrm{div}_x (\varrho \mathbf{u} \otimes \mathbf{u}) - \Delta \eta - \partial_t \, \mathrm{div}_x (\varrho \mathbf{u})$$

$$+ \mathrm{div}_x (\varrho \beta + \eta) \nabla \Phi \mathbf{u}. \tag{7}$$

Since we already know about the space setting for the quantities $\{\varrho, \varrho \mathbf{u}, \varrho \mathbf{u} \otimes \mathbf{u}, \mathbb{S}\}$, relation (7) can be viewed as an elliptic equation to be resolved with respect to the pressure p to obtain an estimate

$$p \in L^r((0,T) \times \Omega) \text{ for } r > 0.$$

The most problematic term is certainly

$$\boxed{\partial_t \Delta^{-1} \operatorname{div}_x(\varrho \mathbf{u})}$$

for which there are no estimates available.
Here, the idea is to use the fact that ϱ is a **renormalized solution** of the continuity equation. In that case, we can use (7) to obtain

$$\int_0^T \int_\Omega \psi p B(\varrho)\ dxdt$$

$$\approx \text{bounded terms} + \int_0^T \int_\Omega \partial_t \psi (\Delta^{-1} \operatorname{div}_x)[\varrho \mathbf{u}] B(\varrho)\, dxdt$$

$$+ \int_0^T \int_\Omega \partial_t \psi (\Delta^{-1} \operatorname{div}_x)[\varrho \mathbf{u}] \partial_t B(\varrho) dxdt,$$

for any $\psi \in \mathcal{D}(0, T)$, where

$$\partial_t B(\varrho) = -b(\varrho)\,\mathrm{div}_x\,\mathbf{u} - \mathrm{div}_x(B(\varrho)\mathbf{u}).$$

In other words, if we succeed to make this formal procedure rigorous, we get pressure estimates of the form

$$pB(\varrho)\,\text{bounded in}\,L^1_{loc}((0,T)\times\Omega)$$

for a suitable function B.

Riesz operator

To this end, we introduce the *Riesz integral operator,* $\mathcal{R}$:

$$\mathcal{R}_i[v](\mathbf{x}) \equiv (-\Delta)^{-1/2}\partial_{x_i} = c\lim_{\varepsilon\to 0}\int_{\varepsilon\le|\mathbf{y}|\le\frac{1}{\varepsilon}} v(\mathbf{x}-\mathbf{y})\frac{y_i}{|\mathbf{y}^{N+1}}d\mathbf{y}.$$

or, equivanelntly, in terms of its Fourier symbol,

$$\mathcal{R}_i(\xi) = \frac{i\xi}{|\xi|},\ i = 1,\ldots,N.$$

Lemma (Calderon and Zygmund Theorem)

The Riesz operator $\mathcal{R}_i, i = 1,\ldots,N$ *defined above is a bounded linear operator on* $L^p(\mathbb{R}^N)$ *for any* $1 < p < \infty$.

Lemma

Let $\{\varrho_\delta, \mathbf{u}_\delta, \eta_\delta\}_{\delta>0}$ be a sequence of artificial pressure solutions. Then, there exists a constant $c(T)$, independent of δ, such that

$$\int_0^T \int_\Omega \varrho_\delta^{\gamma+\theta} \, dxdt \leq c(T),$$

where $\Theta = \min\{\frac{2}{3}\gamma - 1, \frac{1}{4}\}$.
If Ω is unbounded, then $\nabla_x \Phi$ is no longer integrable and we cannot simply apply existing results. To prove the bound in this case, let Δ^{-1} be the inverse Laplacian realized using Fourier multipliers.
For each fixed $\delta > 0$, let the test-function $\mathbf{v}_\delta$ be given as

$$\mathbf{v}_\delta = \nabla \Delta^{-1} \varrho_\delta^\theta.$$

Thus,

$$\mathbf{v}_\delta \in_b L^\infty(0,T;W^{1,s}(\Omega)) \cap L^\infty(0,T;L^\infty(\Omega)).$$

Next, since $(\varrho_\delta, \mathbf{u}_\delta)$ is a renormalized solution to the continuity equations, with $B(\varrho_\delta) = \varrho_\delta^\theta$ states

$$\partial_t \varrho_\delta^\theta = -\operatorname{div}(\varrho_\delta^\theta \mathbf{u}) - (\theta - 1)\varrho_\delta^\theta \operatorname{div}\mathbf{u},$$

in the sense of distributions on $(0,T)\times\Omega$. For notational convenience, we observe that

$$\|\partial_t \mathbf{v}_\delta\|_{L^p(0,T;L^q(\Omega))} = \|\nabla\Delta^{-1}\partial_t \varrho_\delta^\theta\|_{L^p(0,T;L^q(\Omega))}$$

$$\leq \|\varrho_\delta^\theta \mathbf{u}_\delta\|_{L^p(0,T;L^q(\Omega))} + \|\varrho^\theta \operatorname{div}\mathbf{u}\|_{L^p(0,T;L^r(\Omega))},$$

for appropriate $1 \leq p, q \leq \infty$ and $r^* = q$.

Next, we apply $\mathbf{v}_\delta$ as test function for the momentum equation to obtain

$$\int_0^T \int_\Omega a\varrho_\delta^{\gamma+\theta}\ dxdt$$

$$= -\int_0^T \int_\Omega (\varrho_\delta \mathbf{u}_\delta)\partial_t \mathbf{v}_\delta + \varrho_\delta \mathbf{u}_\delta \otimes \mathbf{u}_\delta : \nabla \mathbf{v}_\delta\ dxdt$$

$$+ \int_0^T \int_\Omega \mu \nabla \mathbf{u}_\delta \nabla \mathbf{v}_\delta + \lambda \operatorname{div} \mathbf{u}_\delta \operatorname{div} \mathbf{v}_\delta\ dxdt$$

$$- \int_0^T \int_\Omega \eta_\delta \varrho_\delta^\theta - (\varrho_\delta \beta + \eta_\delta)\nabla \Phi_\delta \mathbf{v}_\delta\ dxdt - \int_\Omega \mathbf{m}_0 \mathbf{v}_\delta(0,\cdot)\ dx$$

$$:= I_1 + I_2 + I_3.$$

Next, we apply $\mathbf{v}_\delta$ as test function for the momentum equation to obtain

$$\begin{aligned}\int_0^T\int_\Omega a\varrho_\delta^{\gamma+\vartheta}\,dxdt = &-\int_0^T\int_\Omega (\varrho_\delta \mathbf{u}_\delta)\partial_t \mathbf{v}_\delta + \varrho_\delta \mathbf{u}_\delta \otimes \mathbf{u}_\delta : \nabla \mathbf{v}_\delta \; dxdt \\ &+ \int_0^T\int_\Omega \mu \nabla \mathbf{u}_\delta \nabla \mathbf{v}_\delta + \lambda \operatorname{div} \mathbf{u}_\delta \operatorname{div} \mathbf{v}_\delta \; dxdt \\ &- \int_0^T\int_\Omega \eta_\delta \varrho_\delta^\theta - (\varrho_\delta \beta + \eta_\delta)\nabla \Phi_\delta \mathbf{v}_\delta \; dxdt \\ := I_1 + I_2 + I_3.\end{aligned}$$

To conclude it remains to bound I_1, I_2, and I_3, independently of δ. We start with the I_1 term:

$$
\begin{aligned}
|I_1| &:= \left| \int_0^T \int_\Omega \varrho_\delta \partial_t \mathbf{v} + \varrho_\delta \mathbf{u}_\delta \otimes \mathbf{u}_\delta : \nabla \mathbf{v}_\delta dx dt \right| \\
&\leq \left\| \varrho_\delta \mathbf{u}_\delta \right\|_{L^\infty(0,T:L^{\frac{2\gamma}{\gamma+1}}(\Omega))} \left\| \varrho_\delta^\theta \mathbf{u}_\delta \right\|_{L^1(0,T;L^{\frac{2\gamma}{\gamma-1}}(\Omega))} \\
&+ \left\| \varrho_\delta \mathbf{u}_\delta \right\|_{L^2(0,T:L^{m_2}(\Omega))} C(T) \left\| \varrho^\theta \operatorname{div} \mathbf{u} \right\|_{L^2(0,T;L^r(\Omega))} \\
&\qquad + \left\| \varrho_\delta \mathbf{u}_\delta \otimes \mathbf{u}_\delta \right\|_{L^2(0,T;L^{c_2}(\Omega))} \left\| \nabla \mathbf{v}_\delta \right\|_{L^2(0,T;L^{c_2'}(\Omega))},
\end{aligned}
$$

where

$$
r = \frac{6\gamma}{3\gamma - 6 + 4\gamma}, \qquad r^* = (m_2)', \qquad c_2' = \frac{3\gamma}{2\gamma - 3} \leq s.
$$

Now, we estimate

$$\left\|\varrho_\delta^\theta \mathbf{u}_\delta\right\|_{L^1(0,T;L^{\frac{2\gamma}{\gamma-1}})} \leq C(T)\|\varrho_\delta^\theta\|_{L^\infty(0,T;L^s(\Omega))}\|\mathbf{u}_\delta\|_{L^2(0,T;L^{2^*}(\Omega))},$$

$$\|\varrho_\delta^\theta \operatorname{div} \mathbf{u}_\delta\|_{L^2(0,T;L^r(\Omega))} \leq C\|\operatorname{div} \mathbf{u}_\delta\|_{L^2(0,T;L^2(\Omega))}\|\varrho_\delta^\theta\|_{L^\infty(0,T;L^s(\Omega))},$$

and hence conclude that

$$|I_1| \leq C(T).$$

Next, we easily deduce the bound

$$|I_2| \leq C(T),$$

and it only remains to bound I_3.

$$\begin{aligned} |I_3| \leq \|\eta_\delta\|_{L^2(0,T;L^{3/2}(\Omega))} \|\varrho^\theta\|_{L^2(0,T;L^{\frac{3}{2}}(\Omega))} \\ + \|(\beta\varrho_\delta + \eta_\delta)\nabla\Phi\|_{L^1(0,T;L^1(\Omega))} \|\mathbf{v}_\delta\|_{L^\infty(0,T;L^\infty(\Omega))} \\ \leq C(T)\left(1 + \|(\beta\varrho_\delta + \eta_\delta)\nabla\Phi\|_{L^\infty(0,T;L^1(\Omega))}\right). \end{aligned}$$

Using the energy estimate and the requirements on the potential, we readily deduce

$$\sup_{t\in(0,T)} \int_\Omega |\beta\varrho_\delta + \eta_\delta||\nabla\Phi| \ dx$$
$$\leq \|\nabla\Phi\|_{L^\infty(B(0,R))} \sup_{t\in(0,T)} \int_{B(0,R)} \beta\varrho_\delta + \eta_\delta \ dx$$
$$+ C \sup_{t\in(0,T)} \int_{\Omega\setminus B(0,R)} (\beta\varrho_\delta + \eta_\delta)\Phi \ dxdt \leq C(T),$$

which brings the proof to an end.

Artificial pressure limit

At the previous step, the key in renormalizing the continuity equation was using the integrability gain from the artificial pressure to ensure that ϱ_ε was bounded in $L^2(0, T; L^2(\Omega))$. Having lost this integrability through the passage $\delta \to 0$ and since we require that $\gamma > \frac{3}{2}$, we proceed by defining the *oscillation defect measure*. our assumption that the pressure is given by $p(\varrho) = \varrho^\gamma$ for $\gamma > \frac{3}{2}$ The oscillations defect measure is defined as

$$\boxed{\mathbf{osc}_p[\varrho_\delta \to \varrho](\mathcal{O}) := \sup_{k \geq 1} \left(\limsup_{n \to \infty} \int_{\mathcal{O}} |T_k(\varrho_\delta) - T_k(\varrho)|^p dvdt \right).}$$

The oscillation defect measure

$$\boxed{\mathbf{osc}_p[\varrho_\delta \to \varrho](\mathcal{O}) := \sup_{k\geq 1}\left(\limsup_{n\to\infty} \int_{\mathcal{O}} |T_k(\varrho_\delta) - T_k(\varrho)|^p dvdt \right).}$$

The functions T_k are cuttoff functions defined by

$$T_k(z) := kT\left(\frac{z}{k}\right)$$

where T is such that for nonnegative arguments, $T(z) = z$ for $z \in [0,1]$, $T(z) = 2$ for $z \geq 3$, and a smooth extension is used over the interval $[0,2]$.

The validity of the weak continuity of the effective viscous pressure implies that the oscillations defect measure is bounded:

$$\mathbf{osc}_{\gamma+1}[\varrho_\delta \to \varrho](\mathcal{O}) \leq c(|\mathcal{O}|).$$

$$\Downarrow$$

$$\varrho_\delta \to \varrho \text{ strongly in } L^1((0,T)\times\Omega).$$

Relative entropy representation for the Navier-Stokes-Smoluchowski system

In the spirit of Dafermos (1979), given an entropy $\mathcal{E}(U)$ we can define the relative entropy by

$$\mathcal{H}(U|\overline{U}) := \mathcal{E}(U) - \mathcal{E}(\overline{U}) - D\mathcal{E}(\overline{U}) \cdot (U - \overline{U}) \tag{8}$$

where D stands for the total differentiation operator with respect to $\varrho, \mathbf{m}$, and η. In the present context,

$$U = \begin{bmatrix} \varrho \\ \mathbf{m} := \varrho \mathbf{u} \\ \eta \end{bmatrix}, \quad \overline{U} = \begin{bmatrix} r \\ \overline{\mathbf{m}} := r\mathbf{U} \\ s \end{bmatrix}$$

and

$$\mathcal{E}(U) := \frac{|\mathbf{m}|^2}{2\varrho} + \frac{a}{\gamma - 1}\varrho^\gamma + \eta \ln \eta + (\beta\varrho + \eta)\Phi. \tag{9}$$

Thus, from the definition, the relative entropy is

$$
\begin{aligned}
\mathcal{H}(U|\overline{U}) &= \frac{|\mathbf{m}|^2}{2\varrho} + \frac{a}{\gamma-1}\varrho^\gamma + \eta\ln\eta + (\beta\varrho+\eta)\Phi \\
&\quad - \frac{|\overline{\mathbf{m}}|^2}{2r} - \frac{a}{\gamma-1}r^\gamma - s\ln s - (\beta r + s)\Phi \\
&\quad - \begin{bmatrix} -\frac{|\mathbf{U}|^2}{2} + \frac{a\gamma}{\gamma-1}r^{\gamma-1} + \beta\Phi \\ \mathbf{U} \\ \ln s + 1 + \Phi \end{bmatrix} \cdot \begin{bmatrix} \varrho - r \\ \varrho\mathbf{u} - r\mathbf{U} \\ \eta - s \end{bmatrix} \\
&= \frac{\varrho|\mathbf{u}|^2}{2} + \frac{a}{\gamma-1}\varrho^\gamma + \eta\ln\eta + \beta\varrho\Phi + \eta\Phi \\
&\quad - \frac{r|\mathbf{U}|^2}{2} - \frac{a}{\gamma-1}r^\gamma - s\ln s - \beta r\Phi - s\Phi \\
&\quad + \frac{\varrho|\mathbf{U}|^2}{2} - \frac{r|\mathbf{U}|^2}{2} - \frac{a\gamma}{\gamma-1}r^{\gamma-1}\varrho + \frac{a\gamma}{\gamma-1}r^\gamma - \beta\varrho\Phi + \beta r\Phi \\
&\quad - \varrho\mathbf{u}\cdot\mathbf{U} + r|\mathbf{U}|^2 - \eta\ln s + s\ln s - \eta + s - \eta\Phi + s\Phi \quad (10)
\end{aligned}
$$

After some basic calculations, the relative entropy is calculated to be

$$\mathcal{H}(U|\overline{U}) = \frac{\varrho}{2}|\mathbf{u}-\mathbf{U}|^2 + \frac{a}{\gamma-1}\left(\varrho^\gamma - r^\gamma\right) - \frac{a\gamma}{\gamma-1}r^{\gamma-1}(\varrho - r)$$
$$+\eta\ln\eta - s\ln s - (\ln s+1)(\eta - s), \tag{11}$$

or equivalently,

$$\mathcal{H}(U|\overline{U}) = \frac{\varrho}{2}|\mathbf{u}-\mathbf{U}|^2 + E_F(\varrho, r) + E_P(\eta, s),$$

where

$$\begin{aligned}
E_F(\varrho, r) &:= H_F(\varrho) - H_F'(r)(\varrho - r) - H_F(r)\\
E_P(\eta, s) &:= H_P(\eta) - H_P'(s)(\eta - s) - H_P(s)\\
H_F(\varrho) &:= \frac{a}{\gamma-1}\varrho^\gamma,\ \ H_P(\eta) := \eta\log\eta,\ \ P_F = H_F',\ P_P = H_P'.
\end{aligned}$$

Letting

$$r = r(t,x),\ \ \mathbf{U} = \mathbf{U}(t,x),\ \ s = s(t,x)$$

be smooth functions on $[0,T] \times \overline{\Omega}$ with $r, s > 0$ on $[0,T] \times \overline{\Omega}$ and

$$\mathbf{U}|_{\partial\Omega} = 0,$$

it is shown in Section 3 that for smooth $\{\varrho, \mathbf{u}, \eta\}$,

$$\int_\Omega \frac{1}{2}\varrho|\mathbf{u} - \mathbf{U}|^2 + E_F(\varrho, r) + E_P(\eta, s)\,dx(\tau)$$

$$+ \int_0^\tau \int_\Omega [\mathbb{S}(\nabla \mathbf{u}) - \mathbb{S}(\nabla \mathbf{U})] : \nabla(\mathbf{u} - \mathbf{U})\,dx\,dt$$

$$\leq \int_\Omega \frac{1}{2}\varrho_0|\mathbf{u}_0 - \mathbf{U}_0|^2 + E_F(\varrho_0, r_0) + E_P(\eta_0, s_0)dx + \int_0^\tau \mathcal{R}(\varrho, \mathbf{u}, \eta, r, \mathbf{U}, s)\,dt \tag{12}$$

where

$$
\begin{aligned}
&\mathcal{R}(\varrho, \mathbf{u}, \eta, r, \mathbf{U}, s) \\
&:= \int_\Omega \operatorname{div}(\mathbb{S}(\nabla \mathbf{U})) \cdot (\mathbf{U} - \mathbf{u})\, dx - \int_\Omega \varrho(\partial_t \mathbf{U} + \mathbf{u} \cdot \nabla \mathbf{U}) \cdot (\mathbf{u} - \mathbf{U}) dx \\
&- \int_\Omega \partial_t P_F(r)(\varrho - r) + \nabla P_F(r) \cdot (\varrho \mathbf{u} - r \mathbf{U})\, dx \\
&- \int_\Omega [\varrho(P_F(\varrho) - P_F(r)) - E_F(\varrho, r)] \operatorname{div} \mathbf{U}\, dx \\
&- \int_\Omega \partial_t P_P(s)(\eta - s) + \nabla P_P(s) \cdot (\eta \mathbf{u} - s \mathbf{U})\, dx \\
&- \int_\Omega [\eta(P_P(\eta) - P_P(s)) - E_P(\eta, s)] \operatorname{div} \mathbf{U}\, dx \\
&- \int_\Omega \nabla (P_P(\eta) - P_P(s)) \cdot (\nabla \eta + \eta \nabla \Phi)\, dx \\
&- \int_\Omega (\beta \varrho + \eta) \nabla \Phi \cdot (\mathbf{u} - \mathbf{U})\, dx - \int_\Omega \frac{\eta \nabla s}{s} \cdot (\mathbf{u} - \mathbf{U})\, dx. \qquad (13)
\end{aligned}
$$

Free energy solutions

$\{\varrho, \mathbf{u}, \eta\}$ is an admissible **free energy solution** of **Problem D**, supplemented with the initial data $\{\varrho_0, \mathbf{u_0}, \eta_0\}$ provided that

- $\varrho \geq 0$, $\mathbf{u}$ is a **renormalized** solution of the continuity equation,that is,

$$\int_0^T \int_\Omega \left(\varrho B(\varrho) \partial_t \varphi + \varrho B(\varrho) \mathbf{u} \cdot \nabla_x \varphi - b(\varrho) \operatorname{div} \mathbf{u} \varphi \right) dxdt$$

$$= - \int_\Omega \varrho_0 B(\varrho_0) \varphi(0, \cdot) dx$$

  holds for any test function $\varphi \in \mathcal{D}([0, T) \times \overline{\Omega})$ and suitable b and B.

- The balance of momentum holds in distributional sense. The velocity field $\mathbf{u}$ belongs to the space $L^2(0, T; W^{1,2}(\Omega; R^3))$, therefore it is legitimate to require $\mathbf{u}$ to satisfy the boundary conditions in the sense of traces.

- $\eta \geq 0$ is a weak solution of the Smoluchowski equation. That is,

$$\int_0^\infty \int_\Omega \eta \partial_t \varphi + \eta \mathbf{u} \cdot \nabla \varphi - \eta \nabla \Phi \cdot \nabla \varphi - \nabla \eta \nabla \varphi dxdt$$

$$= - \int_\Omega \eta_0 \varphi(0, \cdot) dx$$

is satisfied for test functions $\varphi \in \mathcal{D}([0, T) \times \bar{\Omega})$ and any $T > 0$. In particular,

$$\eta \in L^2([0, T]; L^{3/2}(\Omega)) \cap L^1(0, T; W^{1, \frac{3}{2}}(\Omega))$$

- Given the total free-energy of the system by

$$E(\varrho, \mathbf{u}, \eta)(t) := \int_\Omega \left(\frac{1}{2}\varrho|\mathbf{u}|^2 + \frac{a}{\gamma - 1}\varrho^\gamma + \eta \log \eta + (\beta\varrho + \eta)\Phi \right),$$

then $E(\varrho, \mathbf{u}, \eta)(t)$ is finite and bounded by the initial energy of the system

$$E(\varrho, \mathbf{u}, \eta)(t) \leq E(\varrho_0, \mathbf{u}_0, \eta_0) \;\text{ a.e. }\; t > 0$$

Moreover, the following free energy-dissipation inequality holds

$$\int_0^\infty \int_\Omega \left(\mu|\nabla \mathbf{u}|^2 + \lambda|\operatorname{div} \mathbf{u}|^2 + |2\nabla\sqrt{\eta} + \sqrt{\eta}\nabla\Phi|^2\right) \, dt$$

$$\leq E(\varrho_0, \mathbf{u}_0, \eta_0)$$

Definition

$\{\varrho, \mathbf{u}, \eta\}$ is a **weakly dissipative solution** of the NSS system with initial data $\{\varrho_0, \mathbf{u}_0, \eta_0\}$ if and only if

- $\{\varrho, \mathbf{u}, \eta\}$ is a weak solution in the sense of Definition above

$$\begin{aligned}
&\int_\Omega \frac{1}{2}\varrho|\mathbf{u}|^2 + \frac{a}{\gamma-1}\varrho^\gamma + \eta \ln \eta + \eta\Phi \, dx(\tau) \\
&+ \int_0^\tau \int_\Omega \mathbb{S}(\nabla \mathbf{u}) : \nabla \mathbf{u} + |2\nabla\sqrt{\eta} + \sqrt{\eta}\nabla\Phi|^2 dx \, dt \\
&= \int_\Omega \frac{1}{2}\varrho_0|\mathbf{u}_0|^2 + \frac{a}{\gamma-1}\varrho_0^\gamma + \eta_0 \ln \eta_0 + \eta_0\Phi \, dx \\
&- \beta \int_0^\tau \int_\Omega \varrho\mathbf{u} \cdot \nabla\Phi \, dx \, dt. \qquad (14)
\end{aligned}$$

- $\{\varrho, \mathbf{u}, \eta\}$ obeys inequality (12) for any suitably smooth functions $\{r, \mathbf{U}, s\}$.

Existence of Weakly Dissipative Solutions

Theorem (Weakly dissipative solutions)

Let us assume that (Ω, Φ) *satisfy the confinement hypotheses* **(HC)** *with* $\Omega \subset \mathbb{R}$ *a bounded domain of class* $C^{2+\nu}, \nu > 0$. *Suppose the initial data* $\{\varrho_0, \mathbf{u}_0, \eta_0\}$ *satisfy*

$$0 < \varrho_0 \in L^\gamma(\Omega),\ \varrho_0|\mathbf{u}_0|^2 \in L^1(\Omega), and \eta_0 \log \eta_0 \in L^1(\Omega).$$

Then the NSS system has a weakly dissipative solution in the sense of the Definition above.

Approximation Scheme

The weakly dissipative solutions are constructed using a three-level approximation scheme. Specifically, a family of finite dimensional spaces X_n for $n \in \mathbb{N}$ consisting of smooth vector-valued functions on $\overline{\Omega}$ vanishing of $\partial\Omega$ is considered.

$$\partial_t \varrho_n + \operatorname{div}(\varrho_n \mathbf{u}_n) = \varepsilon \Delta \varrho_n \tag{15}$$

$$\partial_t \eta_n + \operatorname{div}(\eta_n \mathbf{u}_n - \eta_n \nabla \Phi) = \Delta \eta_n \tag{16}$$

$$\int_\Omega \partial_t(\varrho_n \mathbf{u}_n) \cdot \mathbf{w} dx = \int_\Omega \varrho_n \mathbf{u}_n \otimes \mathbf{u}_n : \nabla \mathbf{w} + (a\varrho_n^\gamma + \eta_n + \delta \varrho_n^\alpha) \operatorname{div} \mathbf{w} dx$$

$$- \int_\Omega \mathbb{S}(\nabla \mathbf{u}_n) : \nabla \mathbf{w} + \varepsilon \nabla \varrho_n \cdot \nabla \mathbf{u}_n \cdot \mathbf{w} dx - \int_\Omega (\beta \varrho_n + \eta_n) \nabla \Phi \cdot \mathbf{w} dx \tag{17}$$

for any $\mathbf{w} \in X_n$, where X_n is a finite dimensional space and α is suitably large exponent.

Boundary conditions:

$$\nabla \varrho_n \cdot \mathbf{n} = 0 \quad \text{on} \quad (0, T) \times \partial\Omega$$

$$\mathbf{u}_n = \nabla \eta_n \cdot \mathbf{n} + \eta_n \nabla \Phi \cdot \mathbf{n} = 0 \text{ on } (0, T) \times \partial\Omega.$$

For notational simplicity, $\{\varrho_n, \mathbf{u}_n, \eta_n\}$ will denote $\{\varrho_{n,\varepsilon,\delta}, \mathbf{u}_{n,\varepsilon,\delta}, \eta_{n,\varepsilon,\delta}\}$ and $\{\varrho_\varepsilon, \mathbf{u}_\varepsilon, \eta_\varepsilon\}$ will denote $\{\varrho_{\varepsilon,\delta}, \mathbf{u}_{\varepsilon,\delta}, \eta_{\varepsilon,\delta}\}$ Here, $\varepsilon, \delta > 0$ are small and α is an appropriate constant. The approximation scheme is also supplemented by the approximate initial data $\{\varrho_{0,\delta}, \mathbf{m}_{0,\delta}, \eta_{0,\delta}\}$.

The approximate initial data are modifications of the original initial data in that

- $0 < \delta \leq \varrho_{0,\delta} \leq \delta^{-1/2\alpha}$ for all $x \in \Omega$, $\varrho_{0,\delta} \to \varrho_0$ in $L^\gamma(\Omega)$, and $|\{x \in \Omega | \varrho_{0,\delta}(x) < \varrho_0(x)\}| \to 0$ as $\delta \to 0$.
- $\mathbf{m}_{0,\delta}(x)$ is the same as $\mathbf{m}_0(x)$ unless $\varrho_{0,\delta}(x) < \varrho_0(x)$, in which case $\mathbf{m}_{0,\delta}(x) = 0$.
- $0 < \delta \leq \eta_{0,\delta} \leq \delta^{-1/2\alpha}$ for all $x \in \Omega$, $\eta_{0,\delta} \to \eta_0$ in $L^2(\Omega)$, and $|\{x \in \Omega | \eta_{0,\delta}(x) < \eta_0(x)\}| \to 0$ as $\delta \to 0$.

Approximate Relative Entropy Inequality

Strategy.

- We use the approximate difference $\mathbf{u}_n - \mathbf{U}_m$ as a test function in the approximate momentum equation (17).
- This difference and its quandratic form are employed in the construction of the approximate relative entropy functional.
- By monitoring the evolution in time of this functional we obtain first the approximate relative inequality (18), and subsequently, by passing to the limit, we obtain the existence of weakly dissipative solutions.

$$
\begin{aligned}
&\frac{\mathrm{d}}{dt}\int_\Omega \frac{1}{2}\varrho_n|\mathbf{u}_n-\mathbf{U}_n|^2+E_F(\varrho_n,r_m)+E_P(\eta_n,s_m)dx\\
&+\int_\Omega [\mathbb{S}(\nabla\mathbf{u}_n)-\mathbb{S}(\nabla\mathbf{U}_m)]:\nabla(\mathbf{u}_n-\mathbf{U}_m)dx+\frac{\delta}{\alpha-1}\frac{\mathrm{d}}{dt}\int_\Omega \varrho_n^\alpha dx\\
&\leq \int_\Omega \mathrm{div}(\mathbb{S}(\nabla\mathbf{U}_m))\cdot(\mathbf{u}_n-\mathbf{U}_m)dx\\
&-\int_\Omega \varrho_n(\partial_t\mathbf{U}_m+\mathbf{u}_n\cdot\nabla\mathbf{U}_m)\cdot(\mathbf{u}_n-\mathbf{U}_m)dx\\
&-\int_\Omega \partial_t P_F(r_m)(\varrho_n-r_m)+\nabla P_F(r_m)\cdot(\varrho\mathbf{u}-r_m\mathbf{U}_m)dx\\
&-\int_\Omega [\varrho_n(P_F(\varrho_n)-P_F(r_m))-E_F(\varrho_n,r_m)]\,\mathrm{div}\,\mathbf{U}_m dx\\
&-\int_\Omega \partial_t P_P(s_m)(\eta_n-s_m)+\nabla P_P(s_m)\cdot(\eta_n\mathbf{u}_n-s_m\mathbf{U}_m)dx\\
&-\int_\Omega [\eta_n(P_P(\eta_n)-P_P(s_m))-E_P(\eta_n,s_m)]\,\mathrm{div}\,\mathbf{U}_m dx
\end{aligned}
$$

A Weak-Strong Uniqueness Result for the NSS system

Motivated by **Dafermos' theory of Relative Entropy (1979)**, and several recent results by Berthelin and Vasseur (2005), and Mellet and Vasseur (2007), we consider a class of weak solutions satisfying the relative entropy inequality and show that the latter may be used to derive various **weak-strong uniqueness** results in this class of solutions.

Energy Functional and Relative Entropy

$$E(\varrho, r, \eta, s) = E_F(\varrho, r) + E_P(\eta, s)$$

$$E_F(\varrho, r) := \frac{a\varrho^\gamma}{\gamma - 1} - \frac{a\gamma r^{\gamma-1}}{\gamma - 1}(\varrho - r) - \frac{ar^\gamma}{\gamma - 1}$$

$$E_P(\eta, s) := \eta \ln \eta - (\ln s + 1)(\eta - s) - s \ln s$$

where r and s are smooth functions on $[0, T] \times \bar{\Omega}$ s.t.

$$r, s > 0 \,\text{on}\, [0, T] \times \bar{\Omega}$$

and $\mathbf{U}$ is smooth om $[0, T] \times \bar{\Omega}$ and $\mathbf{U}|_{\partial\Omega} = 0$.

Relative Entropy Inequality for the Fluid Particle System

$$
\begin{aligned}
&\int_\Omega \left(\frac{1}{2}\varrho|\mathbf{u}-\mathbf{U}|^2 + E(\varrho,r,\eta,s) + [\beta(\varrho-r)+(\eta-s)]\Phi \right)(\tau,\cdot)dx \\
&\qquad + \int_0^\tau \int_\Omega (\mathbb{S}(\nabla\mathbf{u}) - \mathbb{S}(\nabla\mathbf{U})) : \nabla(\mathbf{u}-\mathbf{U})dxdt \\
&\qquad + \int_0^\tau \int_\Omega |2(\nabla\sqrt{\eta} - \nabla\sqrt{s}) + (\sqrt{\eta}-\sqrt{s})\nabla\Phi|^2 dxdt \\
&\leq \int_\Omega \left(\frac{1}{2}\varrho_0|\mathbf{u}_0-\mathbf{U}_0|^2 + E(\varrho_0,r_0,\eta_0,s_0) + [\beta(\varrho_0-r_0)+(\eta_0-s_0)]\Phi \right) dx \\
&\qquad + \int_0^\tau \mathcal{R}(\varrho,\mathbf{u},\eta,r,\mathbf{U},s)dt
\end{aligned}
$$

for any smooth $r, \mathbf{U}, s$.

Weak-Strong Uniqueness Result

Using the relative entropy inequality, we show that a weak solution to the NSS system with certain extra regularity properties is the same as the **suitable weak solution** with the same initial data.

Moving Domains

Generalized penalty methods. Motivation.

A popular class of methods to deal with **moving domains** are the so-called **generalized penalty methods**.

Incompressible fluids. Consider the Navier-Stokes equations for an incompressible fluid in a domain containing an **obstacle**. Let D be an open set containing the **obstacle** $\tilde{\Omega}$, that is, $\tilde{\Omega} \subset D$ and the set $D \setminus \tilde{\Omega}$ is filled with incompressible fluid.
Strategy: Whereas normally the fluid equations are posed strictly in the fluid domain $D \setminus \tilde{\Omega}$, we instead add a **singular term** to the momentum equation and pose the Navier-Stokes equations over the entire domain D as follows:

$$\begin{cases} \partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} + \nabla P = \Delta \mathbf{u} - \frac{1}{\varepsilon} \mathbf{1}_{\tilde{\Omega}} \mathbf{u}, \\ \operatorname{div} \mathbf{u} = 0 \text{ in } (0, T) \times D. \end{cases} \tag{19}$$

The term $\frac{1}{\varepsilon}\mathbf{1}_{\tilde{\Omega}}\mathbf{u}$ added in the momentum equation is the **penalty term**, where ε is a small parameter tending to zero and $\mathbf{1}_{\tilde{\Omega}}$ denotes the characteristic function of the domain $\tilde{\Omega}$. In the limit as $\varepsilon \to 0$, the penalization forces $\mathbf{u} = 0$ in the obstacle domain $\tilde{\Omega}$ and $\mathbf{u}$ satisfies the standard Navier-Stokes in the true fluid domain $D \setminus \tilde{\Omega}$. Indeed, formally expanding the velocity as

$$\mathbf{u} = \mathbf{u}_0 + \varepsilon \mathbf{u}_1 + O(\varepsilon^2),$$

and plugging into the momentum equation in (19), we can match orders of ε to deduce:

$$O(1/\varepsilon) : \mathbf{1}_{\tilde{\Omega}} u_0 = 0,$$

$$O(1) : \partial_t \mathbf{u}_0 + \mathbf{u}_0 \cdot \nabla \mathbf{u}_0 + \nabla P = \Delta \mathbf{u}_0 - \mathbf{1}_{\tilde{\Omega}} \mathbf{u}_1.$$

This implies the leading order term $\mathbf{u}_0$ vanishes in $\tilde{\Omega}$, satisfies the standard Navier-Stokes equations in $D \setminus \tilde{\Omega}$, and in addition the correction $\mathbf{u}_1$ satisfies the Darcy law in the obstacle domain

$$\mathbf{u}_1 + \nabla P = 0, \text{ in } (0, T) \times \tilde{\Omega}.$$

Global Existence: Moving Domains

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0$$

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho(\mathbf{u} \otimes \mathbf{u}) + \nabla_x(p(\varrho) + \eta) - \mu \Delta \mathbf{u} - \lambda \nabla_x div_x \mathbf{u}$$

$$= -(\eta + \beta \varrho) \nabla \Phi,$$

$$\partial_t \eta + \mathrm{div}(\eta(\mathbf{u} - \nabla \Phi)) - \Delta \eta = 0.$$

$\varrho = \varrho(t, x)$ – total mass density $\quad$ t – time, $x \in \Omega \subset \mathbb{R}^3$
$u = u(t, x)$ – velocity field
$\eta = \eta(t, x)$ – the density of the particles

$$p(\varrho) = a\varrho^{\gamma} \qquad a > 0, \gamma > 1, \beta \neq 0$$

Φ external potential

$$\mu > 0,\ \lambda + \frac{2}{3}\mu \geq 0 \qquad \text{viscosity parameters}$$

$\beta > 0$ if Ω is unbounded

Objective:

The global existence of weak solutions within a **moving domain** Ω_t.

The result is established via a **penalization technique**. The main components of this method are:

- the introduction of a singular term in the momentum equation (the so-called **Brinkman penalization**)
- the penalization of the viscosity.

From a modeling perspective these terms model the solid portions of domain as porous media, with permeability approaching zero.

Strategy:

- Enclose the moving domain within a fixed domain such that the fluid is allowed to flow through solid obstacles.
- **Penalization of the viscosity** is used to get rid of extra shear terms that appear in the solid portion of the domain.
- A **key ingredient** is getting rid of the terms supported on the solid part of the domain.

Assumptions on the spatial domain

- Let $\Omega_0 \subset\subset D \subset \mathbb{R}^3$ denote a domain contained in the fixed domain D, also known as the universal domain. At a later time $t > 0$, the initial domain Ω_0 has moved to the new position Ω_t.
- The family $\{\Omega_t\}_{t=0}^{T}$ then forms a one-parameter transformation of the domain Ω_0. We assume that each image is compactly contained within D. The boundary $\partial\Omega_t$ is denoted by Γ_t.
- The boundary of the domain Ω_t occupied by the fluid and the particles is described by means of a **given** velocity field $\mathbf{V}(t, \mathbf{x})$, where $t \geq 0$ and $\mathbf{x} \in \mathbb{R}^3$.

Assumptions on the boundary

When viewed as a subset of $[0, T] \times D$, moving spatial domains form non-cylindrical space-time domains. In this context, we define the **fluid space-time domain** Q^f by

$$Q^f := \bigcup_{t \in (0,T)} (\{t\} \times \Omega_t).$$

The set

$$Q^s := ((0, T) \times D) \ \overline{Q^f},$$

is often called the **solid domain**.

The evolution of the domain is characterized by a **prescribed** velocity field $\mathbf{V}(t, vcx)$ defined over $(0, T) \times D$. The velocity field allows us to define the position $\mathbf{X}(t, \mathbf{x})$ as

$$\begin{cases} \frac{d}{dt}\mathbf{X}(t, \mathbf{x}) = \mathbf{V}(t, \mathbf{X}(t, \mathbf{x})), \ t > 0, \\ \mathbf{X}(0, \mathbf{x}) = \mathbf{x}, \ \mathbf{x} \in \Omega_0. \end{cases}$$

The domains therefore evolve according to

$$\Omega_t = \mathbf{X}(t,x)(t,\Omega_0).$$

Furthermore, $\mathbf{V}$ is assumed to have the following regularity

$$\mathbf{V} \in C^{2+\nu}([0,T] \times \overline{D}; \mathbb{R}).$$

We also define here the "solid" part of the domain,

$$Q^s = ((0,T) \times D) \backslash \overline{Q^f}.$$

The *no-slip boundary conditions* on the solid wall are imposed

$$\mathbf{u}(t,\cdot)\big|_{\Gamma_t} = \mathbf{V}(t,\cdot)\big|_{\Gamma_t}, \text{ for any } t \geq 0. \tag{20}$$

In addition, a no-flux condition for particle density holds,

$$(\nabla\eta + \eta\nabla\Phi)\cdot\nu = 0 \quad \text{on} \quad (0,T)\times\Gamma_t. \tag{21}$$

The Function Spaces

With an evolving spatial domain, the function spaces in which solutions are looked for need to be modified accordingly. Consider for example the **heat equation**

$$u_t - \Delta u = 0, \ (0, T) \times \Omega,$$

supplemented with Dirichlet boundary data. Weak solutions as typically sought in the Bochner spaces

$$\mathbf{u} \in L^2(0, T; H_0^1(\Omega)), \ \mathbf{u}_t \in L^2(0, T; H^{-1}(\Omega)).$$

The functions $\mathbf{u} : (0, T) \to H_0^1(\Omega)$ therefore are valued in the fixed space $H_0^1(\Omega)$.

Important point: In the case of moving domains, the interval $(0, T)$ should get mapped to the **full range of spaces**, for instance

$$\bigcup(\{t\} \times H_0^1(\Omega_t)).$$

The way to do this is as follows:
"Embed functions into the **global** space and extend by zero outside the moving domains Ω_t."
We define:

$$L^{p,q}(Q^f) \equiv L^p(0,T;L^q(\Omega_t)) :=$$

$$\{\mathbf{u} \in L^p(0,T;L^q(D)) | \mathbf{u}(t,\cdot) = 0 \text{ over } D \setminus \Omega_t \text{ for a.e. } t \in (0,T)\},$$

with the norm

$$\|\mathbf{u}\|_{L^{p,q}(Q^f)} := \begin{cases} \left(\int_0^T \|\mathbf{u}(t)\|^p_{L^q(\Omega_t)} dt \right)^{\frac{1}{p}}, & \text{if } p < \infty, \\ \operatorname{ess\,sup}_{t \in (0,T)} \|\mathbf{u}(t)\|_{L^q(\Omega_t)}, & \text{if } p = \infty. \end{cases}$$

The spaces $L^{p,q}(Q^f)$ are Banach spaces.

Let $l \in \mathbb{N}$, and let α be a multi-index. We define

$$W^l_{p,q}(Q^f) = L^p(0,T;W^{l,q}(\Omega_t)) :=$$

$$\left\{ \mathbf{u} \in L^{p.q}(Q^f) | \partial^\alpha \mathbf{u} \in L^{p,q}(Q^f),\ |\alpha| \leq l \right\}.$$

with the norm

$$\|\mathbf{u}\|_{W^l_{p,q}(Q^f)} := \sum_{|\alpha|\leq l} \|\partial^\alpha \mathbf{u}\|_{L^{p,q}(Q^f)}.$$

The space of functions continuous with respect to the weak-topology of $L^\gamma(\Omega_t)$ is defined by

$$C([0,T];L^\gamma_{wk}(\Omega_t)) :=$$

$$\left\{ \mathbf{u} \in C([0,T];L^\gamma_{wk}(D)) | \mathbf{u}(t,\cdot) = 0 \text{ on } D \setminus \Omega_t \text{ for all } t \in (0,T) \right\}.$$

In order to define the space of test functions we need to make use of the **ALE map** T_t given by

$$T_t(\mathbf{x}) = \mathbf{X}(t, \mathbf{x}), \text{ for all } \mathbf{x} \in \Omega_0.$$

The homeomorphism T_t is used to push the test functions $\mathcal{D}([0, T] \times \Omega_0; \mathbb{R}^N)$ to the set $(0, T) \to \Omega_t$. Define now

$$\mathcal{D}([0, T] \times \Omega_t; \mathbb{R}^N) :=$$

$$\left\{ \mathbf{u} : Q^f \to \mathbb{R} | \mathbf{u}(t, x) = \hat{\mathbf{u}}(t, T_t^{-1}(x)), \hat{\mathbf{u}} \in \mathcal{D}([0, T] \times \Omega_0; \mathbb{R}^N) \right\}$$

Weak formulation

The fluid- particle system is given by

$$
\begin{cases}
\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0 \\
\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho(\mathbf{u} \otimes \mathbf{u}) + \nabla_x(p(\varrho) + \eta) - \mu \Delta \mathbf{u} - \lambda \nabla_x \mathrm{div}_x \mathbf{u} \\
\qquad = -(\eta + \beta \varrho)\nabla \Phi, \\
\partial_t \eta + \mathrm{div}(\eta(\mathbf{u} - \nabla \Phi)) - \Delta \eta = 0.
\end{cases}
\tag{22}
$$

The system is posed on the space-time domain Q^f.
The no-slip boundary conditions are imposed on the velocity,

$$
\mathbf{u}(t,\cdot)\big|_{\Gamma_t} = \mathbf{V}(t,\cdot)\big|_{\Gamma_t}, \text{ for any } t \geq 0. \tag{23}
$$

In addition, a no-flux condition for particle density holds,

$$
(\nabla \eta + \eta \nabla \Phi) \cdot \nu = 0 \quad \text{on} \quad (0, T) \times \Gamma_t. \tag{24}
$$

Initial data are prescribed such that

$$\begin{cases} \varrho_0 \in L^\gamma(D), \ \ \varrho_0 \geq 0 \text{ a.e. in}\, \Omega_0. \\ \eta_0 \in L^1(D), \ \ \eta_0 \geq 0 \text{ a.e. in}\, \Omega_0. \\ \mathbf{m}_0 \in L^1(D;\mathbb{R}^3), \ \ \frac{\mathbf{m_0}}{\varrho_0} \in L^1(D), \end{cases} \tag{25}$$

and all initial data is assumed to vanish on $D \setminus \Omega_0$.

Main Result: Theorem.

Theorem

Let $\Omega_0 \subset\subset D \subset \mathbb{R}^3$ be a bounded domain with boundary of class $C^{2+\nu,}$ $\nu > 0$. Assume that the pressure p is given by

$$p(\varrho, \eta) = \varrho^\gamma + \eta, \text{ with } \gamma > 3/2.$$

Let $\mathbf{V}$ be a given vector field belonging to $C^{2+\nu}([0,T] \times \overline{D}; \mathbb{R}^3)$,

$$\mathbf{V}|_{\partial D} = 0.$$

Suppose that the initial data satisfy (25) and all initial data is assumed to vanish on $D \setminus \Omega_0$. Then, there exists a weak solution $(\varrho, \mathbf{u}, \eta)$, of Problem (22).

Strategy

The main ingredients of our approach can be formulated as follows:

- For the construction of a suitable approximating scheme **penalizing** the boundary behavior, extra diffusion and viscosity terms are introduced in the weak formulation. The central component of this approach is the addition of a singular term

$$\int_0^T \int_{\Omega_t} \frac{\chi(\mathbf{u} - \mathbf{V})}{\varepsilon} \cdot \varphi \, dxdt, \quad \varepsilon > 0 \text{ small}, \tag{26}$$

  in the momentum equation. This extra term models solid obstacles as porous media, with porosity and viscous permeability approaching zero. Effectively, the problem is reformulated over a fixed domain such that the fluid is allowed to "flow" through solid obstacles.

- In addition to (26), we introduce **variable** shear viscosity coefficients $\mu = \mu_\omega$ and $\lambda = \lambda_\omega$, vanishing outside the fluid domain and remaining positive within the fluid domain, to take care of extra stress terms that appear in the **solid** domain.

- In constructing the approximating problem we employ a number of ingredients: a parameter δ which enables us to introduce an **artificial pressure** essential for the establishment of suitable pressure estimates and parameters ε and ω for the penalization of the boundary behavior and viscosity. Keeping δ, ϵ, ω fixed, we solve the modified problem in a (bounded) reference domain $D \subset \mathbb{R}^3$ chosen in such a way that

  $$\Omega_t \subset \overline{\Omega_t} \subset D \text{ for any } t \geq 0.$$

  Letting $\delta \to 0$ we obtain the solution $(\varrho, \mathbf{u}, \eta)_{\omega,\varepsilon}$ within the fixed reference domain.

- We take the initial densities (ϱ_0, η_0) vanishing outside Ω_0, and letting the penalization $\varepsilon \to 0$ we obtain a "two-phase" model consisting of the **fluid region** and the **solid region** separated by impermeable boundary. We show that the densities vanish in the "solid" part of the reference domain, specifically on $((0, T) \times D) \setminus \overline{Q^f}$.
- The penalization ε is taken to vanish and then we perform the limit $\omega \to 0$.

Penalization scheme

Denote by $\chi = \chi(t, x)$ the characteristic function of Q^s, that is,

$$\chi(t,x) = \begin{cases} 0 & \text{if } \; t \in (0,T), x \in \Omega_t \\ 1 & \text{otherwise} \end{cases} \tag{27}$$

This function is used to separate the fluid and "solid" domains and represents a weak solution to the transport equation

$$\left\{ \begin{array}{l} \partial_t \chi + \mathbf{V} \cdot \nabla \chi = 0 \\ \chi(0, \cdot) = \mathbb{1}_D - \mathbb{1}_{\Omega_0}. \end{array} \right. \tag{28}$$

Penalization Scheme

Fix a reference spatial domain $D \subset \mathbb{R}^3$ containing Ω_0. System (22) is replaced by a penalized problem

$$\partial_t \varrho + \mathrm{div}(\varrho \mathbf{u}) = 0, \tag{29}$$

$$\begin{aligned} \partial_t(\varrho \mathbf{u}) + \mathrm{div}(\varrho \mathbf{u} \otimes \mathbf{u}) \ &+ \nabla(p(\varrho) + \eta) - \mu_\omega \Delta \mathbf{u} - \lambda_\omega \nabla_x \mathrm{div}_x \mathbf{u} = \\ &-(\eta + \beta\varrho)\nabla\Phi - \frac{1}{\varepsilon}\chi(\mathbf{u} - \mathbf{V}) \end{aligned} \tag{30}$$

$$\partial_t \eta + \mathrm{div}(\eta(\mathbf{u} - \nabla\Phi)) - \Delta_x \eta = 0. \tag{31}$$

considered in the cylinder $(0, T) \times D$.

The penalized problem is supplemented with the boundary conditions

$$\mathbf{u}\big|_{\partial D} = \mathbf{V}\big|_{\partial D} = 0, \tag{32}$$

$$(\nabla\eta + \eta\nabla\Phi)\cdot\nu\big|_{\partial D} = 0 \tag{33}$$

with ν denoting the outer normal vector to the boundary ∂D, and initial conditions

$$\varrho(0,\cdot) = \varrho_{0,\varepsilon} \geq 0, \ \ (\varrho\mathbf{u})(0,\cdot) = (\varrho\mathbf{u})_{0,\varepsilon}, \ \ \eta(0,\cdot) = \eta_{0,\varepsilon} \geq 0, \tag{34}$$

such that

$$\varrho_{0,\varepsilon} \to \varrho_0 \ \text{in} \ L^\gamma(D), \ \varrho_0|_{\Omega_0} > 0, \ \varrho_0|_{D \setminus \Omega_0} = 0, \tag{35}$$

$$(\varrho \mathbf{u})_{0,\varepsilon} \to (\varrho \mathbf{u})_0 \ \text{in} \ L^1(D; \mathbb{R}^3), \ (\varrho \mathbf{u})_0|_{D \setminus \Omega_0} = 0, \tag{36}$$

$$\int_D \frac{|(\varrho \mathbf{u})_{0,\varepsilon}|^2}{\varrho_{0,\varepsilon}} \, dx < c, \tag{37}$$

$$\eta_{0,\varepsilon} \to \eta_0 \ \text{in} \ L^2(D), \ \eta_0|_{\Omega_0} > 0, \ \eta_0|_{D \setminus \Omega_0} = 0, \tag{38}$$

where c is independent of $\varepsilon \to 0$.

In order to eliminate extra stresses that appear we introduce a variable shear viscosity coefficient $\mu = \mu_\omega(t, \mathbf{x})$ where, $\mu = \mu_\omega$ remains strictly positive in Q^f but vanishes in the complement as $\omega \to 0$, namely μ_ω is taken such that

$$\mu_\omega \in C_c^\infty\left([0, T] \times \mathbb{R}^3\right), \ \ 0 < \underline{\mu}_\omega \leq \mu_\omega(t, \mathbf{x}) \leq \mu \text{ in } [0, T] \times D,$$

$$\mu_\omega = \begin{cases} \mu = \text{const} > 0 & \text{in } Q^f \\ \mu_\omega \to 0 & \text{a.e. in } ((0, T) \times D) \setminus Q^f \end{cases}$$

We penalize the coefficient $\lambda = \lambda_\omega(t, x)$ exactly the same way.
Finally we modify the initial data

$$(\varrho \mathbf{u})_{0,\varepsilon,\omega} = \frac{|(\varrho \mathbf{u})_{0,\varepsilon,\omega}|^2}{\varrho_{0,\varepsilon,\omega}} = 0, \ \text{ whenever } \ \varrho_{0,\varepsilon,\omega} = 0.$$

The weak formulation of the penalized problem reads.

Weak formulation

[Free energy solutions of the penalized problem]

Assume that (D, Φ) satisfies the confinement hypotheses **(HC)**.
We say that
$\{\varrho, \mathbf{u}, \eta\}$ is a free-energy solution of problem (29)-(31) with initial and boundary data satisfying (32)-(34) respectively provided that the following hold:

- $\rho_{\epsilon,\omega} \geq 0$ represents a renormalized solution of equation (31) on a time-space cylinder $(0, \infty) \times D$, that is, for any test function $\varphi \in \mathcal{D}([0, T) \times \overline{D})$, any $T > 0$, and any b such that

$$b \in L^\infty \cap C[0, \infty), \ B(\varrho) = \varrho B(1) + \varrho \int_1^\varrho \frac{b(z)}{z^2} \, \mathrm{d}z,$$

the following integral identity holds:

$$\int_0^\infty \int_D \left(B(\varrho)\partial_t\varphi + B(\varrho)\mathbf{u}\cdot\nabla\varphi - b(\varrho)\,\mathrm{div}\,\mathbf{u}\varphi\right)\,\mathrm{d}x\mathrm{d}t \tag{39}$$

$$= \int_D B(\varrho)(T,\cdot)\varphi(T,\cdot)dx - \int_D B(\varrho_0)\varphi(0,\cdot)\,dx$$

The density, velocity, and momentum are required to have the following regularity

$$\varrho \in L^\infty(0,T;L^\gamma(D)),\ \ \varrho \geq 0 \,\text{a.e. in}\,(0,T)\times D,$$

$$\mathbf{u} \in L^2(0,T;W_0^{1,2}(D,\mathbb{R}^3))$$

$$\varrho\mathbf{u} \in L^\infty(0,T;L^{2\gamma/(\gamma-1)}(D;\mathbb{R}^3))$$

- The balance of momentum holds in distributional sense, namely

$$\int_0^\infty \int_D \Big(\varrho \mathbf{u} \cdot \partial_t \varphi + \varrho \mathbf{u} \otimes \mathbf{u} : \nabla \varphi + (p(\varrho) + \eta) \operatorname{div} \varphi \Big) dxdt =$$

$$\int_0^\infty \int_D (\mu \nabla \mathbf{u} \ + \lambda \operatorname{div} \mathbf{u} \mathbb{I}) : \nabla \varphi - (\eta + \beta \varrho) \nabla \Phi \cdot \varphi \, dxdt$$

$$+ \int_0^\infty \int_D \frac{\chi(\mathbf{u} - \mathbf{V})}{\varepsilon} \cdot \varphi \, dxdt \qquad (40)$$

$$+ \int_D (\varrho \mathbf{u})(T, \cdot) \cdot \varphi(T, \cdot) dx - \int_D (\varrho \mathbf{u})_0 \cdot \varphi(0, \cdot) \, dx$$

for any test function $\varphi \in \mathcal{D}([0, T); \mathcal{D}(\overline{D}; \mathbb{R}^3))$ and any $T > 0$ satisfying $\varphi|_{\partial D} = 0$.

All quantities appearing in (40) are supposed to be at least integrable. In particular, the velocity field

$$\mathbf{u} \in L^2(0, T; W^{1,2}(D; \mathbb{R}^3)),$$

therefore it is legitimate to require $\mathbf{u}_{\epsilon,\omega}$ to satisfy the boundary conditions (32) in the sense of traces.

- The integral identity

$$\int_0^\infty \int_D \eta \partial_t \varphi + \eta \mathbf{u}_{\varepsilon,\omega} \cdot \nabla \varphi - \eta \nabla \Phi \cdot \nabla \varphi - \nabla \eta \cdot \nabla \varphi \, dxdt$$

$$= \int_D \eta(T,\cdot)\varphi(T,\cdot) dx - \int_D \eta_0 \varphi(0,\cdot)\, dx \tag{41}$$

is satisfied for test functions $\varphi \in \mathcal{D}([0,T) \times \overline{D})$ and any $T > 0$. All quantities appearing in (41) must be at least integrable on $(0,T) \times D$. In particular,

$$\eta_{\epsilon,\omega} \in L^2(0,T; W^{1,1}(D)) \cap L^1(0,T; W^{1,\frac{3}{2}}(D)).$$

$$\eta \geq 0 \, \text{a.e. in} \, (0,T) \times D.$$

- Given the total free-energy of the system by

$$E(\varrho, \mathbf{u}, \eta)(t) :=$$

$$\int_D \left(\frac{1}{2} \varrho |\mathbf{u}|^2 + \frac{a}{\gamma - 1} \varrho^\gamma + \eta \log \eta + (\beta \varrho + \eta) \Phi \right) dx,$$

$E(\varrho, \mathbf{u}, \eta)(t)$ is finite and the following free energy dissipation inequality holds

$$E(\varrho, \mathbf{u}, \eta)(\tau)$$

$$+ \int_0^\tau \int_D (\mu |\nabla \mathbf{u}|^2 + \lambda |\operatorname{div} \mathbf{u}|^2 + |2\nabla \sqrt{\eta} + \sqrt{\eta} \nabla \Phi|^2) \; dxdt$$

$$\leq E(\varrho_0, \mathbf{u}_0, \eta_0) - \int_0^\tau \int_D \frac{\chi}{\varepsilon} (\mathbf{u} - \mathbf{V}) \cdot \mathbf{u} dxdt \qquad (42)$$

Modified energy inequality

Choosing as a test function

$$\varphi = \psi_n(t)\mathbf{V}, \psi_n \in C_c^\infty[0, T), \psi_n \to \mathbb{1}_{[0,\tau)}$$

in (40) and adding to the inequality (42), we find that

Modified energy

$$
\begin{aligned}
&\int_D \left(\frac{1}{2} \varrho_{\varepsilon,\omega} |\mathbf{u}_{\varepsilon,\omega}|^2 + \frac{a}{\gamma - 1} \varrho^{\gamma}_{\varepsilon,\omega} + \eta_{\varepsilon,\omega} \log \eta_{\varepsilon,\omega} + (\beta \varrho_{\varepsilon,\omega} + \eta_{\varepsilon,\omega}) \Phi \right) (\tau, \cdot) dx \\
&+ \int_0^\tau \int_D \left(\mu_\omega |\nabla \mathbf{u}_{\varepsilon,\omega}|^2 + \lambda |\operatorname{div} \mathbf{u}_{\varepsilon,\omega}|^2 + |2 \nabla \sqrt{\eta_{\varepsilon,\omega}} + \sqrt{\eta_{\varepsilon,\omega}} \nabla \Phi|^2 \right) \ dxdt \\
&+ \frac{1}{\varepsilon} \int_0^\tau \int_D \chi |\mathbf{u}_{\varepsilon,\omega} - \mathbf{V}|^2 \, dxdt \\
&\leq \int_D \left(\frac{1}{2} \frac{|(\varrho \mathbf{u})_{0,\varepsilon,\omega}|^2}{\varrho_{0,\varepsilon,\omega}} + \frac{a}{\gamma - 1} \varrho^{\gamma}_{0,\varepsilon,\omega} + (\eta \log \eta)_{0,\varepsilon,\omega} + (\beta \varrho + \eta)_{0,\varepsilon,\omega} \Phi \right) dx \\
&+ \int_D (\varrho_{\epsilon,\omega} \mathbf{u}_{\epsilon,\omega} \cdot \mathbf{V})(\tau, \cdot) - (\varrho \mathbf{u})_{0,\varepsilon,\omega} \cdot \mathbf{V}(0, \cdot) dx \\
&+ \int_0^\tau \int_D \mathbb{S}_{\varepsilon,\omega} : \nabla \mathbf{V} - \varrho_{\epsilon,\omega} \mathbf{u}_{\epsilon,\omega} \cdot \partial_t \mathbf{V} - \varrho_{\epsilon,\omega} \mathbf{u}_{\epsilon,\omega} \otimes \mathbf{u}_{\epsilon,\omega} : \nabla \mathbf{V} \\
&- (\eta_{\epsilon,\omega} + \beta \rho_{\epsilon,\omega}) \nabla_x \Phi \cdot \mathbf{V} - \left(\frac{a}{\gamma - 1} \varrho^{\gamma}_{\epsilon,\omega} + \eta_{\epsilon,\omega} \right) \operatorname{div} \mathbf{V} \, dxdt
\end{aligned} \tag{43}
$$

Uniform bounds

The modified energy inequality yields uniform bounds on $(\rho_{\epsilon,\omega}, \mathbf{u}_{\epsilon,\omega}, \eta_{\epsilon,\omega})$ independent of $\varepsilon \to 0$ provided $\mathbf{V}$ is sufficiently smooth, namely

$$\{\sqrt{\varrho_{\varepsilon,\omega}}\mathbf{u}_{\varepsilon,\omega}\}_{\{\varepsilon,\omega>0\}} \text{ bounded in } L^\infty(0,T;L^2(D;\mathbb{R}^3)) \tag{44}$$

$$\{\varrho_{\varepsilon,\omega}\}_{\{\varepsilon,\omega>0\}} \text{ bounded in } L^\infty(0,T;L^\gamma(D)) \tag{45}$$

$$\{\nabla\mathbf{u}_{\varepsilon,\omega}\}_{\{\varepsilon,\omega>0\}} \text{ bounded in } L^2(0,T;L^2(D;\mathbb{R}^3\times\mathbb{R}^3)) \tag{46}$$

$$\{\mathrm{div}\,\mathbf{u}_{\varepsilon,\omega}\}_{\{\varepsilon,\omega>0\}} \text{ bounded in } L^2(0,T;L^2(D)) \tag{47}$$

$$\{\nabla\sqrt{\eta_{\varepsilon,\omega}}\}_{\{\varepsilon,\omega>0\}} \text{ bounded in } L^2(0,T;L^2(D;\mathbb{R}^3)) \tag{48}$$

In addition,

$$\int_0^\tau \int_D \chi|\mathbf{u}_{\varepsilon,\omega} - \mathbf{V}|^2 dx\,dt = \int_{Q^s} |\mathbf{u}_{\varepsilon,\omega} - \mathbf{V}|^2\,dxdt \leq \varepsilon c, \tag{49}$$

for a.a. $\tau \in (0,T)$ with c independent of ε, ω, where we used the definition of $\chi(t,x)$.

Using the embedding of $W^{1,2}(D)$ in $L^6(D)$ (since $D \subset \mathbb{R}^3$) on the last bound listed above, it is clear that $\{\eta_{\varepsilon,\omega}\}_{\{\varepsilon,\omega>0\}} \in_b L^1(0,T;L^3(D))$. This, and mass conservation implies

$$\{\eta_{\varepsilon,\omega}\}_{\{\varepsilon,\omega>0\}} \in_b L^1(0,T;L^3(D)) \cap L^\infty(0,T;L^1(D)). \qquad (50)$$

Using this result, and that

$$2\nabla\sqrt{\eta} = \frac{\nabla\eta}{\sqrt{\eta}},$$

it is also clear that

$$\{\eta_{\varepsilon,\omega}\}_{\{\varepsilon,\omega>0\}} \in_b L^1(0,T;W^{1,\frac{3}{2}}(D)) \cap L^2(0,T;W^{1,1}(D)). \qquad (51)$$

By Poincaré's inequality and (46), we get that

$$\{\mathbf{u}_{\varepsilon,\omega}\}_{\{\varepsilon,\omega>0\}} \text{ bounded in } L^2(0,T;W_0^{1,2}(D;\mathbb{R}^3)). \qquad (52)$$

Pressure estimates and pointwise convergence of the fluid density

The detailed analysis yields the estimates needed to deal with the nonlinear pressure, $p(\rho) = a\rho^{\gamma}$, obtain pointwise convergence of the fluid density ρ, and pass to the limit in (39), (40). In particular,

$$\int_K p(\varrho_{\varepsilon,\omega})\varrho_{\varepsilon,\omega}^{\nu}\, dxdt \leq c(K) \text{ for any compact } K \subset Q^f, \tag{53}$$

and these estimates can be extended up to the boundary, and

$$\rho_{\varepsilon,\omega} \to \rho_{\omega} \text{ in } L^q((0,T) \times D) \text{ for any } 1 \leq q < \gamma.$$

Convergence in the set Q^s

The convergence of the densities in the "solid" part of the domain play a crucial role in the analysis. Establishing that,

$$\varrho(t,x) = 0 \quad \text{for a.a. } (t,x) \in Q^s$$

relies on regularizing the equation of continuity and employing the commutator lemma of DiPerna and Lions. It remains to show that

$$\eta(t,x) = 0 \quad \text{for a.a. } (t,x) \in Q^s.$$

The cutoff function $\chi(t,x)$ satisfies the transport equation (28). In anticipation of using a suitable (smooth) test function, consider instead the unique function $\overline{\chi} \in C^\infty(\mathbb{R}^3)$ solving

$$\partial_t \overline{\chi} + \mathbf{V} \cdot \nabla_x \overline{\chi} = 0 \quad t > 0, x \in \mathbb{R}^3,$$

with the initial data satisfying

$$C^{\infty}(\mathbb{R}^3) \ni \overline{\chi}(0,\cdot) = \begin{cases} > 0 & x \in D \backslash \Omega_0 \\ < 0 & x \in \Omega_0 \cup (\mathbb{R}^3 \backslash \bar{D}) \end{cases}, \quad \nabla_x \overline{\chi}_0 \neq 0 \quad \text{on } \partial\Omega_0.$$

We define the level-set test function,

$$\varphi_\xi = \begin{cases} 1 & \overline{\chi} \geq \xi \\ \dfrac{\overline{\chi}}{\xi} & 0 \leq \overline{\chi} < \xi \\ 0 & \overline{\chi} < 0 \end{cases} = \min\left\{\frac{\overline{\chi}}{\xi}, 1\right\}^+, \tag{54}$$

supported on $D \backslash \Omega_\tau$, see [**?**], [**?**].

Lemma

Let

$$\begin{cases} \eta_{\varepsilon,\omega} \in L^2(0,T;W^{1,1}(D)) \cap L^1(0,T;W^{1,\frac{3}{2}}(D)), \eta_{\varepsilon,\omega} \geq 0, \\ \mathbf{u}_{\varepsilon,\omega} \in L^2(0,T;W^{1,2}(D;\mathbb{R}^3)) \end{cases}$$

be a weak solution of the Smoluchowski equation that is, (41)*, holds for all* $\varphi \in \mathcal{D}([0,T) \times \overline{D})$ *and any* $T > 0$*. Let the initial data satisfy*

$$\eta_0 \in L^2(D) \cap L^1_+(D), \quad \eta_0\big|_{D\setminus\Omega_0} = 0.$$

Then for $\xi > 0$ *and* $\overline{\chi}$ *defined as above, it holds that*

$$\lim_{\xi\to 0} \frac{1}{\xi} \int_0^\tau \int_{\{0\leq\bar{\chi}<\xi\}} (\eta\nabla_x\Phi + \nabla_x\eta) \cdot \nabla_x\bar{\chi}\, dxdt = 0, \tag{55}$$

for any $\tau > 0$.

Proof.

Plugging (54) into (41) and rearranging we get that

$$\frac{1}{\xi}\int_0^\tau\int_{\{0\le\overline{\chi}<\xi\}}(\eta_{\varepsilon,\omega}\nabla_x\Phi+\nabla_x\eta_{\varepsilon,\omega})\cdot\nabla_x\overline{\chi}\,dxdt = \frac{1}{\xi}\int_0^\tau\int_{\{0\le\overline{\chi}<\xi\}}\eta_{\varepsilon,\omega}(\mathbf{u}_{\varepsilon,\omega}-\mathbf{V})\cdot\nabla_x\overline{\chi}\,dxdt+\int_D\eta_{0,\varepsilon,\omega}\varphi_\xi(0,\cdot)\,dx. \tag{56}$$

Since we can pass $\varepsilon,\omega\to 0$ on the left side in (56), it suffices to show that right side vanishes as we take $\varepsilon,\omega\to 0$ and $\xi\to 0$ successively. First,

$$\lim_{\varepsilon,\omega\to 0}\int_D\eta_{0,\varepsilon,\omega}\varphi_\xi(0,\cdot)dx=\int_{\Omega_0}\eta_0\varphi_\xi(0,\cdot)dx=0,$$

since on Ω_0, we have $\overline{\chi}(0,\cdot)<0$ and so $\varphi_\xi(0,\cdot)=0$.

Now,

$$\lim_{\varepsilon,\omega\to 0} \frac{1}{\xi} \int_0^\tau \int_{\{0\leq\overline{\chi}<\xi\}} \eta_{\varepsilon,\omega}(\mathbf{u}_{\varepsilon,\omega} - \mathbf{V}) \cdot \nabla_x \overline{\chi}\, dxdt$$

$$= \frac{1}{\xi} \int_0^\tau \int_{\{0\leq\overline{\chi}<\xi\}} \eta(\mathbf{u} - \mathbf{V}) \cdot \nabla_x \overline{\chi}\, dxdt = 0,$$

since $\mathbf{u} = \mathbf{V}$ a.e. in $D\backslash\Omega_0$, i.e. where $\overline{\chi} \geq 0$, using (**??**). Letting $\xi \to 0$ concludes the proof of the lemma.

Lemma

Under the same conditions as lemma 12, the following holds,

$$\eta(\tau, \cdot)|_{D\setminus\Omega_\tau} = 0 \quad \textit{for a.a. } \tau \in [0, T].$$

Proof.

First note that by choosing a test function having the form

$$\varphi_n = \psi_n(t)\varphi(t,x), \varphi \in C_c^\infty([0,T) \times \bar{D}), \psi_n \to \mathbb{1}_{[0,\tau)} \text{ as } n \to \infty,$$

and $\psi_n \in C^\infty[0,T)$, we can rewrite the weak form (**??**) as

$$\int_D \eta_{\varepsilon,\omega}(\tau,\cdot)\varphi(\tau,\cdot) - \eta_{0,\varepsilon,\omega}\varphi(0,\cdot)\, dx \tag{57}$$

$$= \int_0^\tau \int_D \eta_{\varepsilon,\omega}(\partial_t\varphi + \mathbf{u}_{\varepsilon,\omega}\cdot\nabla_x\varphi) - (\eta_{\varepsilon,\omega}\nabla_x\Phi + \nabla_x\eta_{\varepsilon,\omega})\cdot\nabla_x\varphi\, dxdt,$$

for any $\varphi \in C_c^\infty([0,T) \times \bar{D})$.

It suffices to establish that

$$\int_{D\setminus\Omega_\tau} \eta(\tau,\cdot)\,dx = 0, \quad a.a\ \tau \in (0,T).$$

Inserting φ_ξ into (57), using the initial conditions, and letting $\varepsilon, \omega \to 0$ yields,

$$\int_D \eta(\tau,\cdot)\varphi_\xi(\tau,\cdot)\,\mathrm{d}x = \tag{58}$$

$$\frac{1}{\xi}\int_0^\tau \int_{\{0\le\overline{\chi}<\xi\}} \eta(\mathbf{u}-\mathbf{V})\cdot\nabla_x\overline{\chi} - (\eta\nabla_x\Phi + \nabla_x\eta)\cdot\nabla_x\overline{\chi}\,dxdt.$$

Since $\varphi_\xi(\tau,\cdot) \to \mathbb{1}_{D\setminus\Omega_\tau}$ as $\xi \to 0$ in any $L^p(D), p < \infty$, and $\eta \in L^2(0,T; L^{3/2}(D))$, the left-hand side of (58) converges to

$$\int_{D\setminus\Omega_\tau} \eta(\tau,\cdot)\,\mathrm{d}x,$$

as $\xi \to 0$. Finally, using Lemma 12 and that $\mathbf{u} = \mathbf{V}$ for any $\xi > 0$, it is clear the right hand side of (58) vanishes as $\xi \to 0$.

Singular limits: The limit $\omega \to 0$

Performing the limit $\varepsilon \to 0$, we arrive at the weak formulation of the momentum satisfied, except for the following term

$$\int_0^\infty \int_D (\mu_\omega \nabla_x \mathbf{u}_\omega + \lambda_\omega \mathrm{div}_x \mathbf{u}_\omega \mathbb{I}) : \nabla_x \varphi \, dxdt. \tag{59}$$

Using the viscosity penalization (39) (similarly for λ_ω), vanishing in $((0, T) \times D) \backslash Q^f$ and using that $\mathbf{u}_\omega = \mathbf{V}$ here, we conclude that

$$\int_0^T \int_{D \backslash \Omega_t} (\mu_\omega \nabla_x \mathbf{u}_\omega + \lambda_\omega \mathrm{div}_x \mathbf{u}_\omega \mathbb{I}) : \nabla_x \varphi \, dxdt \to 0 \text{ as } \omega \to 0.$$

Remark.

In fact when letting $\varepsilon, \omega \to 0$ in the momentum equation, the penalization term remains as a weak limit,

$$\frac{\mathbf{u}_{\varepsilon,\omega} - \mathbf{V}}{\varepsilon} \rightharpoonup h \text{ in } L^1(Q^s),$$

This term, which appears artificially in the solid domain, is then removed in the weak formulation by proper choice of test functions.

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