Lec 2: The system of polyconvex elastodynamics

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Outline

The equations of elasticity

Requirements from mechanics – Embedding to a symmetrizable system

Variational approximation schemes

Existence of mv-solutions - transport identities

Uniqueness of smooth solutions within class of entropy mv-solutions

Other approximations

Relaxation

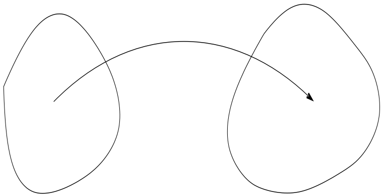
Equations of radial elasticity

Variational schemes preserving the positivity of determinants

The equations of elasticity

$$\frac{\partial^2 y}{\partial t^2} = \operatorname{div} \frac{\partial W}{\partial F}(\nabla y)$$

motion	$y(x,t)$
velocity	$v = \dfrac{\partial y}{\partial t}$
deformation gradient	$F = \nabla y$



balance of mass	$\rho_0 = \rho \det F$
balance of momentum	$\rho_0 \dfrac{\partial^2 y}{\partial t^2} = \operatorname{div} S + \rho_0 b$

Hyperelastic $S = \frac{\partial W}{\partial F}(F)$

$W(F)$ stored energy

- Material frame indifference

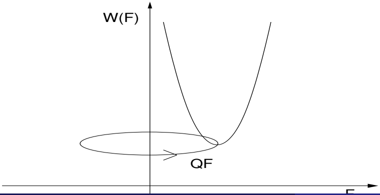
$$W(QF) = W(F) \quad \forall\, Q \in \mathcal{O}^3$$

- Realizibility of mechanical motions

avoid interpenetration of matter

at least positivity of the Jacobian

$$\det F > 0$$



$$W(F) \to \infty \quad \text{as} \quad \det F \to 0$$

It is too restrictive to take $W(F)$ convex

$$\partial_t F_{i\alpha} = \partial_\alpha v_i$$
$$\partial_t v_i = \partial_\alpha \frac{\partial W}{\partial F_{i\alpha}}(F)$$
$$\partial_\alpha F_{i\beta} - \partial_\beta F_{i\alpha} = 0$$

Energy identity

$$\partial_t \left(\frac{1}{2}|v|^2 + W(F) \right) + \partial_\alpha \left(v_i \frac{\partial W}{\partial F_{i\alpha}} \right) = 0$$

Hyperbolicity $\iff$ $W(F)$ is rank-1 convex

$$\iff \frac{\partial^2 W}{\partial F_{i\alpha} \partial F_{j\beta}}(F)\, \xi_i \xi_j \nu_\alpha \nu_\beta > 0 \quad \forall \xi \neq 0\,,\ \nu \in \mathcal{S}^2$$

wave speeds ($d = 3$) are
$$\lambda_1 = \dots \lambda_6 = 0$$
$$\lambda_7 \dots \lambda_{12} = \pm \sqrt{\text{e.v. of accoustic tensor}}$$

Objective

Conservation law theory is intricately connected to notion of convexity
What can be said regarding dynamics when the assumption of convexity is relaxed ?

The Euler-Lagrange equations of the minimization problem

$$\min_{y \in W^{1,\infty}} I[y] = \int_\Omega W(\nabla y)\, dx$$

are the system of elastostatics $\partial_{x_\alpha} \frac{\partial W}{\partial F_{i\alpha}}(\nabla y) = 0$

Critical points of the target problem for the functional

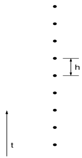
$$J[y] = \int_0^T \int_\Omega \frac{1}{2}|y_t|^2 - W(\nabla y) dxdt$$

provide solutions of the elastodynamics system

$$\partial_t^2 y_i = \partial_{x_\alpha} \frac{\partial W}{\partial F_{i\alpha}}(\nabla y)$$

The functional J is indefinite

$$\partial_t^2 y = \partial_x W'(y_x) = -\frac{\delta}{\delta y}\Big(\int W(y_x) dx \Big)$$



time-step discretization: iterates y^j solve

$$\frac{y^j - 2y^{j-1} + y^{j-2}}{h^2} = \partial_x W'(y_x^j)$$

Constructed via the variational problem

$$\min_{\frac{u-u^0}{h} = v_x} \int_I \frac{1}{2}(v - v^0)^2 + W(u)\, dx$$

where $v = \frac{\delta y}{h}$, $u = y_x$.

a Scheme emerges from a marching algorithm rather than a target variational problem

b Scheme produces entropy weak solution for dimension $d = 1$

Demoulini-Stuart-T. '00

$$\min_{y \in W^{1,\infty}} I[y] = \int_\Omega W(\nabla y)\, dx$$

$W(F)$ is rank-1 convex

strong ellipticity of the E-L equations : $\partial_{x_\alpha} \dfrac{\partial W}{\partial F_{i\alpha}}(\nabla y) = 0$

$W(F)$ is quasiconvex if

$$\int_\Omega W(F + \nabla\phi(x))\, dx \geq W(F)\, |\Omega| \qquad \forall F \in M^{3\times 3}\,,\ \phi \in C_c^\infty(\Omega)$$

equivalent to w.l.s.c. of $I[y] = \int_\Omega W(\nabla y)\, dx$ in $W^{1,\infty}$

$$\nabla y_n \rightharpoonup \nabla y \quad \Longrightarrow \quad \int_\Omega W(\nabla y)\, dx \leq \liminf \int_\Omega W(\nabla y_n)\, dx$$

$W(F)$ is polyconvex

$$W(F) = g(F, \operatorname{cof} F, \det F) = g \circ \Phi(F) \qquad \text{with } g(\Xi) \text{ convex}$$

The integrand $\Phi(F)$ is a null-Lagrangean iff

$$\int_\Omega \Phi(\nabla y + \nabla\phi)\, dx = \int_\Omega \Phi(\nabla y) dx \quad \forall\, y \in W^{1,p}\,,\ \phi \in C_c^\infty$$

$\Phi(F)$ is null-Lagrangean

$$\begin{aligned} \iff &\quad \int_\Omega \Phi(F + \nabla\phi)\, dx = \Phi(F)\,|\Omega| \quad \forall\, F \in M^{3\times 3}\,,\ \phi \in C_c^\infty \\ \iff &\quad \Phi(F) = \alpha\, F + \beta \operatorname{cof} F + c \det F \\ \iff &\quad \partial_\alpha \left(\frac{\partial \Phi}{\partial F_{i\alpha}}(\nabla y) \right) = 0 \quad \text{in } \mathcal{D}' \end{aligned}$$

If $\Phi(\nabla y)$ is null-Lagrangean then it is weakly continuous in $W^{1,p}$.

J. Ball 77, J. Ericksen 62

$$W(F) = g(F, \operatorname{cof} F, \det F) = g \circ \Phi(F)$$

with $g(\Xi)$ a convex function

role of polyconvexity in elastostatics:

- w.l.s.c. in $W^{1,\infty}$ of

$$I[y] = \int_\Omega g \circ \Phi(\nabla y) dx$$

  emerges from convexity of $g(\Xi)$ and the weak continuity of $\Phi(\nabla y)$

- In polyconvex class one achieves existence of minimizers for potentials satisfying

$$W(F) \to \infty \quad \text{as } \det F \to 0$$

Null-Lagrangeans

$$\partial_\alpha \left(\frac{\partial \Phi}{\partial F_{i\alpha}} (\nabla y) \right) = 0 \quad \text{in } \mathcal{D}'$$

Transport identities

$$\partial_t F_{i\alpha} = \partial_\alpha v_i$$
$$\partial_t \Phi^A(F) = \frac{\partial \Phi^A}{\partial F_{i\alpha}} \partial_\alpha v_i = \partial_\alpha \left(\frac{\partial \Phi^A}{\partial F_{i\alpha}} v_i \right) \quad A = 1, ..., 19$$

explicitly

$$\frac{\partial}{\partial t} \det F = \frac{\partial}{\partial x^\alpha} \big((\operatorname{cof} F)_{i\alpha} v_i \big)$$
$$\frac{\partial}{\partial t} (\operatorname{cof} F)_{k\gamma} = \frac{\partial}{\partial x^\alpha} (\epsilon_{ijk} \epsilon_{\alpha\beta\gamma} F_{j\beta} v_i)$$

T. Qin 98

The augmented elasticity system

Elasticity with transport identities; variables (v, F)

$$\partial_t v_i = \partial_\alpha \left(\frac{\partial g}{\partial \Xi^A}(\Phi(F)) \frac{\partial \Phi^A}{\partial F_{i\alpha}}(F) \right)$$
$$\partial_t \Phi(F)^A = \partial_\alpha \left(\frac{\partial \Phi^A}{\partial F_{i\alpha}}(F)\, v_i \right).$$

Symmetrized elasticity system; variables (v, Ξ)

$$\partial_t v_i = \partial_\alpha \left(\frac{\partial g}{\partial \Xi^A}(\Xi) \frac{\partial \Phi^A}{\partial F_{i\alpha}}(F) \right)$$
$$\partial_t \Xi^A = \partial_\alpha \left(\frac{\partial \Phi^A}{\partial F_{i\alpha}}(F)\, v_i \right).$$

both subject to the propagating constraint - involutions

$$\partial_\alpha F_{i\beta} - \partial_\beta F_{i\alpha} = 0$$

Properties of the extension

(a) Elastodynamics is viewed as a constrained evolution:

$$\Xi(\cdot, 0) = \Phi(F(\cdot, 0)) \quad \Longrightarrow \quad \Xi(\cdot, t) = \Phi(F(\cdot, t)) \;\; \forall t$$

(b) The enlarged system admits a strictly convex entropy

$$\eta(v, \Xi) = \frac{1}{2}|v|^2 + g(\Xi)$$

and is thus symmetrizable

$$\partial_t \left(\frac{1}{2}|v|^2 + g(\Xi) \right) - \partial_\alpha \left(v_i \frac{\partial g(\Xi)}{\partial \Xi^A} \frac{\partial \Phi^A(F)}{\partial F_{i\alpha}} \right) = 0$$

Relative entropy in conservation laws - Stability

system of conservation laws $\quad \partial_t u + \operatorname{div} F(u) = 0$

with convex entropy

u entropy (approximate) solution $\quad \partial_t \eta(u) + \operatorname{div} q(u) = -\mu \leq 0$

$\bar{u}$ smooth (conservative) solution $\quad \partial_t \eta(\bar{u}) + \operatorname{div} q(\bar{u}) = 0$

relative entropy $\quad \eta(u|\bar{u}) = \eta(u) - \eta(\bar{u}) - \nabla\eta(\bar{u})(u - \bar{u})$

$$\begin{aligned} \partial_t \eta(u|\bar{u}) + \operatorname{div} q(u|\bar{u}) &= -\mu - \Big(\nabla^2 \eta(\bar{u}) \partial_x \bar{u}\Big) F(u|\bar{u}) \\ &\leq O(1)|u - \bar{u}|^2 \end{aligned}$$

Dafermos 79, DiPerna 79, ...

Based on

$\eta - q$ is entropy - flux pair $\quad \nabla^2\eta(u)\nabla F(u) = \nabla F(u)^T \nabla^2 \eta(u)$

$\eta(u)$ convex

$|f(u|\bar{u})| \leq C\eta(u|\bar{u})$

small

$\nu = \nu_{(x,t)}$, $U = < \nu, \lambda >$ is mv entropy solution of system of conservation laws

$$\partial_t U + \text{div } < \nu_{(x,t)}, f(\lambda) > = 0$$
$$\partial_t < \nu_{(x,t)}, \eta(\lambda) > + \text{div } < \nu_{(x,t)}, q(\lambda) > = -\mu_{x,t} \leq 0$$

relative entropy $\quad \eta(u|\bar{u}) = \eta(u) - \eta(\bar{u}) - \nabla\eta(\bar{u})(u - \bar{u})$

satisfies

$$\partial_t < \nu_{x,t}, \eta(\lambda|\bar{u}) > + \text{ div } < \nu_{x,t}, q(\lambda|\bar{u}) > \leq \ O(1) < \nu_{x,t}, |\lambda - \bar{U}(x,t)|^2 >$$

provides control of the variance $\int |\lambda - \bar{U}(x,t)|^2 d\nu_{x,t}(\lambda)$

uniqueness of smooth within entropic mv solutions

Brenier-DeLellis-Szekelyhidi 11 , Demoulini-Stuart-AT 11

Relative entropy for polyconvex elasticity

(v, F) approximate solution

$$\partial_t F_{i\alpha} = \partial_\alpha v_i$$
$$\partial_t v_i = \partial_\alpha \frac{\partial}{\partial F_{i\alpha}}(g \circ \Phi(F)) + \varepsilon \triangle v_i$$

$(\bar{v}, \bar{F})$ smooth solution of

$$\partial_t F_{i\alpha} = \partial_\alpha v_i$$
$$\partial_t v_i = \partial_\alpha \frac{\partial}{\partial F_{i\alpha}}(g \circ \Phi(F))$$

relative entropy $\eta((v, F)|(\bar{v}, \bar{F})) = \frac{1}{2}|v - \bar{v}|^2 + g\left(\Phi(F)|\Phi(\bar{F})\right)$

relative flux $q_\alpha((v, F)|(\bar{v}, \bar{F})) = \big(\frac{\partial g}{\partial \Xi^A}(\Phi(F)) - \frac{\partial g}{\partial \Xi^A}(\Phi(\bar{F}))\big)(v_i - \bar{v}_i)\frac{\partial \Phi^A}{\partial F_{i\alpha}}(F)$

$$\partial_t \eta_{rel} + \operatorname{div} q_{rel} + \varepsilon |\nabla(v - \bar{v})|^2 = O(1)|\Phi(F) - \Phi(\bar{F})|^2 + O(\varepsilon)$$

Convergence as $\varepsilon \to 0$ to smooth solutions of polyconvex elasticity when g is strictly convex. Convergence in the norm:

$$\int |v - \bar{v}|^2 + |\Phi(F) - \Phi(\bar{F})|^2$$

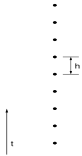
Dafermos, Lattanzio-AT 06

Variational approximation 3-d

$$\frac{\partial^2 y_i}{\partial t^2} = \partial_\alpha \frac{\partial W}{\partial F_{i\alpha}}(\nabla y)$$

Time-step discretization - step size h

$$\frac{y_i - 2y_i^0 + y_i^{(-1)}}{h^2} = \partial_\alpha \frac{\partial W}{\partial F_{i\alpha}}(\nabla y)$$



Question:

> To construct a stable approximation scheme, variational in nature, marching in time.
>
> It has to be energy conservative on strong solutions and energy dissipative on shocks.

Euler-Lagrange equations for the minimization problem

$$\min \int_\Omega \frac{|y_i - 2y_i^0 + y_i^{(-1)}|^2}{2h^2} + W(\nabla y) dx$$

Open problem: Whether for $W(F)$ quasiconvex the scheme decreases the

$W(F) = g \circ \Phi(F)$ polyconvex

$$
\begin{aligned}
\partial_t v_i &= \partial_\alpha \left(\frac{\partial g}{\partial \Xi^A}(\Xi) \frac{\partial \Phi^A}{\partial F_{i\alpha}}(F) \right) \\
\partial_t \Xi^A &= \partial_\alpha \left(\frac{\partial \Phi^A}{\partial F_{i\alpha}}(F)\, v_i \right) .
\end{aligned}
$$

the symmetrized elasticity system and null-Lagrangians suggest the implicit-explicit iterative scheme

$$
\begin{aligned}
\frac{v_i^J - v_i^{J-1}}{h} &= \frac{\partial}{\partial x^\alpha} \left(\frac{\partial g}{\partial \Xi^A}(\Xi^J)\, \frac{\partial \Phi^A}{\partial F_{i\alpha}}(F^{J-1}) \right) \\
\frac{(\Xi^J - \Xi^{J-1})^A}{h} &= \frac{\partial}{\partial x^\alpha} \left(\frac{\partial \Phi^A}{\partial F_{i\alpha}}(F^{J-1}) v_i^J \right) .
\end{aligned}
$$

Iterates (v, Ξ) are constructed by solving the constrained variational problem
Given v^0, $\Xi^0 = (F^0, Z^0, w^0)$,

$$\min \int_{\mathbb{T}^3} \left(\frac{1}{2} |v - v^0|^2 + g(F, Z, w) \right) dx$$

over the affine subspace

$$\mathcal{C} := \Big\{ (v, F, Z, w) : \mathbb{T}^3 \to \mathbb{R}^{22} \text{ subject to the constraints}$$
$$\frac{1}{h}(F_{i\alpha} - F^0_{i\alpha}) = \partial_\alpha v_i \, ,$$
$$\frac{1}{h}(Z_{k\gamma} - Z^0_{k\gamma}) = \partial_\alpha (\epsilon_{ijk} \epsilon_{\alpha\beta\gamma} F^0_{j\beta} v_i) \, ,$$
$$\frac{1}{h}(w - w^0) = \partial_\alpha ((\operatorname{cof} F^0)_{i\alpha} v_i) \qquad \Big\}.$$

Iterates decrease the mechanical energy, obey bounds

$$\sup_j \int_{\mathbb{T}^3} |v^j|^2 + g(\Xi^j) \, dx + \sum_j |v^j - v^{j-1}|^2_{L^2_x} + |\Xi^j - \Xi^{j-1}|^2_{L^2_x} \le E_0$$

Under coercivity for g and bounds for g and $\frac{\partial g}{\partial \Xi}$ we have

$$v^h \rightharpoonup v \quad \text{wk in } L^2$$
$$(F^h, Z^h, w^h) \rightharpoonup (F, Z, w) \quad \text{wk in } L^p \times L^q \times L^r$$

and (v, F) is a measure-valued solution of elasticity, which satisfies the weak from of the geometric transport identities

Demoulini-Stuart-AT 01

Uniqueness of classical solutions for polyconvex elasticity within the class of measure-valued solutions

Demoulini-Stuart-AT 11

Let $(\bar{v}, \bar{F})$ be a smooth solution of elasticity defined on $[0, T]$, $T < T^*$. The approximation scheme converges for $T < T^*$

$$\int |v^h - \bar{v}|^2 + g\left(\Xi^h \middle| \Phi(\bar{F})\right)\, dx \le \mathcal{E}_{(\nabla \bar{v}, \nabla \bar{F})}\Big(v_0^h - \bar{v}_0, \Xi_0^h - \bar{\Xi}_0\Big) + O(h)$$

Miroshnikov-AT 14

Existence of dissipative mv-solutions

Under coercivity for g and bounds for g and $\frac{\partial g}{\partial \Xi}$ we obtain a Young measure ν and a nonnegative concentration measure $\gamma(dxdt)$ such that

$$v^h \rightharpoonup v \quad \text{wk in } L^2$$

$$(F^h, Z^h, w^h) \rightharpoonup (F, \operatorname{cof} F, \det F) \quad \text{wk in } L^p \times L^q \times L^r$$

where $F = \langle \nu, \lambda_F \rangle$, $v = \langle \nu, \lambda_v \rangle$ satisfy

$$\partial_t v_i - \partial_\alpha \Big\langle \nu, \frac{\partial G}{\partial \Xi^A}(\lambda_\Xi) \frac{\partial \Phi^A}{\partial F_{i\alpha}}(\lambda_F) \Big\rangle = 0$$

$$\partial_t \Phi^A(F) - \partial_\alpha \Big(\frac{\partial \Phi^A}{\partial F_{i\alpha}}(F) v_i \Big) = 0$$

and

$$\iint \frac{d\theta}{dt} \Big(\langle \nu, \eta \rangle + \gamma \Big) \, dxdt + \int \theta(0) \eta_0(x) dx \geq 0 \,,$$

for test functions $\theta(t) \geq 0$

i.e, (v, F) is a dissipative measure-valued solution of elasticity, which satisfies the weak from of the geometric transport identities

Let ν, γ, v, F be a dissipative measure valued solution and $\hat{v}$, $\hat{F} \in W^{1,\infty}$ a Lipschitz solution. Then

$$\int \langle \nu, \eta\big((\lambda_v, \lambda_F)|(\bar{v}, \bar{F})\big)\rangle \, dx \leq c_1 \Big(\int \eta((v_0, F_0)|(\bar{v}_0, \bar{F}_0)) \, dx \Big) \; e^{c_2 t}$$

$$\text{where} \quad \eta((v,F)|(\bar{v},\bar{F})) = \frac{1}{2}|v - \bar{v}|^2 + g\left(\Phi(F)|\Phi(\bar{F})\right)$$

Based on an averaged relative entropy calculation and using the weak form of the transport identities and the null-Lagrangean property.

Uniqueness of classical solutions for polyconvex elasticity within the class of dissipative mv-solutions:

If $v_0 = \hat{v}_0$ and $F_0 = \hat{F}_0$ then

$$(v, F) = (\hat{v}, \hat{F})\,, \quad \nu = \delta_{\hat{v}(x,t),\hat{F}(x,t)}\,, \quad \gamma = 0 \text{ on } Q_T$$

Lattice approximation of one dimensional elastodynamics by a spring-mass system

- each atom has identical mass $\varepsilon\rho = \frac{2\pi}{N}\rho$ (total mass $2\pi\rho$)
- Potential energy by $V = \sum_{i=0}^{N-1} W(\frac{x_{i+1}-x_i}{\varepsilon})$, with W strictly convex
- Lagrangian

$$L = T - V = \sum_{i=0}^{N-1} \frac{\varepsilon\rho}{2}\dot{x}_i^2 - \varepsilon W(\frac{x_{i+1}-x_i}{\varepsilon}).$$

$$\begin{aligned} v_i &= \dot{x}_i \\ \rho\frac{dv_i}{dt} &= \frac{1}{\varepsilon}\left(W'(\frac{x_{i+1}-x_i}{\varepsilon}) - W'(\frac{x_i - x_{i-1}}{\varepsilon}) \right) \end{aligned}$$

Set

$$Y(t, X_i) = x_i(t) \quad \text{where } X_i = i\varepsilon$$

then formally Y satisfies the nonlinear wave equation

$$Y_{tt} = \partial_x W'(Y_x)$$

Introduce approximate solution y^ε by piecewise linear and piecewise constant interpolation

Uniform bounds $\int (y_t^\varepsilon)^2 + W(y_x^\varepsilon) dx \leq C$

$y^\varepsilon \rightharpoonup y$ induces a measure-valued solution that is conservative

Convergence before shock formation is an application of the relative entropy method, measure valued wk versus strong uniqueness.

Approximation is dispersive, so the relation of the two systems beyond shock formation is an outstanding open problem

$$\partial_t v_i - \partial_\alpha \left(\tau^A \frac{\partial \Phi^A}{\partial F_{i\alpha}} \right) = 0$$

$$\partial_t F_{i\alpha} - \partial_\alpha v_i = 0$$

$$\partial_t \left(\tau^A - \frac{\partial \sigma_I}{\partial \Xi^A}(\Phi(F)) \right) = -\frac{1}{\varepsilon} \left(\tau^A - \frac{\partial \sigma_E}{\partial \Xi^A}(\Phi(F)) \right)$$

Limit as $\varepsilon \to 0$ polyconvex elasticity system

$$\partial_t v_i - \partial_\alpha \left(\frac{\partial \sigma_E}{\partial \Xi^A}(\Phi(F)) \frac{\partial \Phi^A}{\partial F_{i\alpha}} \right) = 0$$

$$\partial_t F_{i\alpha} - \partial_\alpha v_i = 0$$

This relaxation system is equipped with a globally defined entropy (free energy) with Ψ convex

$$\frac{1}{2}|v|^2 + \Psi(\Phi(F), \tau)$$

Relative entropy yields stability theory for the relaxation limit for a gas: $W(F) = g(\det F) := e(\frac{1}{w}) \circ \det F$ where $e(\rho)$ internal energy

in Lagrangean coordinates

$$\partial_t v_i - \partial_\alpha \left(\left[-p_I(\frac{1}{\det F}) + \tau^A \right] \operatorname{cof} F_{i\alpha} \right) = 0$$
$$\partial_t F_{i\alpha} - \partial_\alpha v_i = 0$$
$$\partial_t \tau^A = -\frac{1}{\varepsilon} \left(\tau^A + p_E(\frac{1}{\det F}) - p_I(\frac{1}{\det F}) \right)$$

in Eulerian coordinates — $\rho = \frac{1}{\det F}$

$$\partial_t \rho + \partial_j (\rho v_j) = 0$$
$$\partial_t (\rho v_i) + \partial_j (\rho v_i v_j) = \partial_j \left([-p_I(\rho) + \tau] \delta_{ij} \right)$$
$$\partial_t (\rho \tau) + \partial_j (\rho v_j \tau) = -\frac{1}{\varepsilon} \rho \left[\tau - p_I(\rho) + p_E(\rho) \right]$$

Additive decomposition of pressures to instantaneous and equilibrium pressures (but not of the Piola-Kirchhoff stresses)

System has a global thermodynamical structure, globally defined entropy that dissipates

Radial isotropic elasticity

$$\frac{\partial^2 y}{\partial t^2} = \operatorname{div} \frac{\partial W}{\partial F}(\nabla y)$$

- Radial motions $\qquad y(x,t) = w(R,t)\frac{x}{R}, \ R = |x|, \ x \in \mathbb{R}^3$
- $$W(F) = \Phi(w_R, \frac{w}{R}, \frac{w}{R})$$

  isotropic material Φ symmetric function and polyconvex, *e.g.*

  $$\Phi = \frac{1}{2}(v_1^2 + v_2^2 + v_3^2) + h(v_1 v_2 v_3) \quad \text{with } h(\delta) \to +\infty \text{ as } \delta \to 0+$$

$$w_{tt} = \frac{1}{R^2}\,\partial_R\left(R^2 \frac{\partial \Phi}{\partial v_1}\big(w_R, \frac{w}{R}, \frac{w}{R}\big)\right) - \frac{1}{R}\Big(\frac{\partial \Phi}{\partial v_2} + \frac{\partial \Phi}{\partial v_3}\Big)\big(w_R, \frac{w}{R}, \frac{w}{R}\big)$$

- To represent a physically realizable motion: $\det F > 0$ with $F = \nabla y$.

$$\det F = w_R (w/R)^2 > 0$$

Null-Lagrangians: Potential energies $\Psi(v_1, v_2, v_3; R)$ for which the functional

$$I[w] = \int_0^1 \Psi\big(\big(w_R, \frac{w}{R}, \frac{w}{R}\big); R\big)\, dR$$

has variational derivative zero: $\Psi = v_1,\ v_1 v_2 R,\ v_1 v_3 R,\ v_1 v_2 v_3 R^2$

Euler-Lagrange identities

$$-\partial_R \Psi_{,1} + R^{-1}\left(\Psi_{,2} + \Psi_{,3}\right) = 0 \quad \forall w\,.$$

transport identities

$$\partial_t \Psi = \partial_R \left(\Psi_{,1}\, v\right)$$

It is possible to write (more than one) extended systems for polyconvex radial elasticity that achieve a convex entropy

Question: Is it possible to construct a variational approximation scheme for radial polyconvex elasticity that preserves the positivity of Jacobian determinants and is consistent with entropy dissipation.

Change of variables: $\rho = R^3$, $\alpha = w^3$. In the new variables $\det F = \alpha_\rho$

Extended system:

$$\begin{cases} \partial_t v = \partial_\rho \Big(3\rho^{2/3} G_{,i}(\Xi)\, \Omega^i_{,1}(\Gamma) \Big) - \rho^{-1/3} G_{,i}(\Xi)\, \big(\Omega^i_{,2}(\Gamma) + \Omega^i_{,3}(\Gamma)\big) \\ \partial_t \beta = \partial_\rho (3v) \\ \partial_t \alpha = 3\alpha^{2/3} v \\ \partial_t \gamma = 2\alpha^{1/3} v \\ \\ \alpha(1) = \lambda, \ \alpha \geq 0, \ \ \alpha_\rho > 0, \ \ (\rho, t) \in (0,1) \times [0, \infty), \end{cases}$$

where $\beta = \frac{w_R}{R^2}$ and $\gamma = w^2$, $\Omega^i(\Gamma)$ the null-Lagrangeans in the new variables.

$$\text{minimize} \quad I(\alpha,\beta,\gamma,v) = \int_0^1 \frac{1}{2}(v-v_0)^2 + G(\Xi)\,d\rho$$
$$= \int_0^1 \frac{1}{2}(v-v_0)^2 + \ldots + h(\alpha_\rho)\,d\rho$$

over the set of admissible functions

$$\mathcal{A}_\lambda = \Big\{(\alpha,\beta,\gamma,v) \in X : \alpha(0) \geq 0,\ \alpha(1) = \lambda,\ \alpha' > 0 \text{ a.e. } \textit{and}$$
$$(1) \qquad I(\alpha,\beta,\gamma,v) < \infty,\ \frac{(\beta-\beta_0)}{h} = 3v',$$
$$\frac{(\alpha-\alpha_0)}{h} = 3{\alpha_0}^{2/3}v,\ \frac{(\gamma-\gamma_0)}{h} = 2{\alpha_0}^{1/3}v\Big\}.$$

The differential constraints in (1) are affine, the condition $\alpha(1) = \lambda$ corresponds to the imposed boundary condition $y(x) = \lambda x$, while $\alpha' > 0$ secures the positivity of determinants

existence of minimizers
Euler-Lagrange equations are satisfied
regularity of minimizers

the iterates satisfy the constraint of positive Jacobian and invertibility

the iterates satisfy a discrete version of entropy inequality

$$\frac{\left(\frac{v^2}{2} + G(\Xi)\right) - \left(\frac{{v_0}^2}{2} + G(\Xi^0)\right)}{h} - \frac{d}{d\rho}\left(3\rho^{2/3} G_{,i}(\Xi)\Omega^i_{,1}(\Gamma^0) v\right) \leq 0$$

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